MULTIPHASE WEAKLY NONLINEAR GEOMETRIC OPTICS FOR SCHRÖDINGER EQUATIONS

RÉMI CARLES, ERIC DUMAS, AND CHRISTOF SPARBER

ABSTRACT. We describe and rigorously justify the nonlinear interaction of highly oscillatory waves in nonlinear Schrödinger equations, posed on Euclidean space or on the torus. Our scaling corresponds to a weakly nonlinear regime where the nonlinearity affects the leading order amplitude of the solution, but does not alter the rapid oscillations. We consider initial states which are superpositions of slowly modulated plane waves, and use the framework of Wiener algebras. A detailed analysis of the corresponding nonlinear wave mixing phenomena is given, including a geometric interpretation on the resonance structure for cubic nonlinearities. As an application, we recover and extend some instability results for the nonlinear Schrödinger equation on the torus in negative order Sobolev spaces.

1. INTRODUCTION

1.1. **Physical motivation.** The (cubic) *nonlinear Schrödinger equation* (NLS)

$$(1.1) \qquad i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^2 u,$$

with $\lambda \in \mathbb{R}^*$, is one of the most important models in nonlinear science. It describes a large number of physical phenomena in nonlinear optics, quantum superfluids, plasma physics or water waves, see e.g. [30] for a general overview. Independent of its physical context one should think of (1.1) as a description of nonlinear waves propagating in a dispersive medium. In the present work we are interested in describing the possible resonant interactions of such waves, often referred to as *wave mixing*. The study of this nonlinear phenomena is of significant mathematical and physical interest: for example, in the context of fiber optics, where (1.1) describes the time-evolution of the (complex-valued) electric field amplitude of an optical pulse, it is known that the dominant nonlinear process limiting the information capacity of each individual channel is given by four-wave mixing, cf. [16, 32]. Due to its cubic nonlinearity, (1.1) seems to be a natural candidate for the investigation of this particular wave mixing phenomena. Similarly, four wave mixing appears in the context of plasma physics where NLS type models are used to describe the propagation of Alfvén waves [28]. Moreover, recent physical experiments have shown the possibility of matter-wave mixing in *Bose–Einstein condensates* [12]. A formal theoretical treatment, based on the Gross–Pitaevskii equation (*i.e.* a cubic NLS describing the condensate wave function in a mean-field limit), can be found in [31, 17]. Finally, we also want to mention the closely related studies on so-called

auto-resonant solutions of NLS given in [13] where again wave mixing phenomena are used as a method of excitation and control of multi-phase waves.

Due to the high complexity of the problem most of the aforementioned works are restricted to the study of small amplitude waves, representing, in some sense, the lowest order nonlinear effects in systems which can approximately be described by a linear superposition of waves. In addition a slowly varying amplitude approximation is usually deployed. By doing so one restricts himself to resonance phenomena which are *adiabatically stable* over large space- and time-scales. We shall follow this approach by introducing a small parameter $0 < \varepsilon \ll 1$, which represents the microscopic/macroscopic scale ratio, and consider a rescaled version of (1.1):

$$(1.2) \qquad i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda\varepsilon|u^\varepsilon|^2 u^\varepsilon.$$

This is a *semi-classically scaled* NLS [6] representing the time evolution of the wave field $u^\varepsilon(t,x)$ on macroscopic length- and time-scales. In the following we seek an asymptotic description of u^ε as $\varepsilon \to 0$ on space/time-intervals, which are independent of ε. Note that due to the small parameter ε in front of the nonlinearity, we consider a *weakly nonlinear regime*. This means that the nonlinearity does not affect the geometry of the propagation, see §1.2 below. Technically, it does not show up in the eikonal equation, but only in the transport equations determining the modulation of the leading order amplitudes. In view of these remarks, the sign of λ (focusing or defocusing nonlinearity) turns out to be irrelevant.

1.2. **A general formal computation.** In order to describe the appearance of the wave mixing in solutions to (1.6), we follow the *Wentzel-Krammers-Brillouin* (WKB) approach, as first rigorously settled by Lax [23]. Consider approximate solutions of (1.2) in the form of high-frequency wave packets, such as

$$(1.3) \qquad a(t,x)e^{i\phi(t,x)/\varepsilon}.$$

For such a single mode to be an approximate solution, it is necessary that the rapid oscillations are carried by a phase ϕ which solves the eikonal equation (see [6], where also other regimes, in terms of the size of the coupling constant, are discussed):

$$(1.4) \qquad \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 = 0.$$

Nonlinear interactions of high frequency waves are then found by considering superpositions of wave packets (1.3). By the cubic interaction, three phases ϕ_1, ϕ_2 and ϕ_3 generate

$$\phi = \phi_1 - \phi_2 + \phi_3.$$

The corresponding term is relevant at leading order if and only if this new phase ϕ is *characteristic*, *i.e.* solves the eikonal equation (1.4) while also each ϕ_j, $j = 1, 2, 3$ does so. More generally, we will have to construct a set of phases $\{\phi_j\}_{j\in J}$, for some index set $J \subset \mathbb{Z}$, such that each ϕ_j is characteristic, and the set is *stable* under the nonlinear interaction. That is, if $k, \ell, m \in J$ are such that $\phi = \phi_k - \phi_\ell + \phi_m$ is characteristic, then $\phi \in \{\phi_j\}_{j\in J}$. Given some index $j \in J$, the set of (four-wave) resonances leading to the phase ϕ_j is then

$$I_j = \big\{(k, \ell, m) \in J^3 \ ; \ \phi_k - \phi_\ell + \phi_m = \phi_j\big\}.$$

One of the tasks of this work to study the structure of I_j. A first important step is obtained by plugging $\phi = \phi_k - \phi_\ell + \phi_m$ into (1.4), since then, an easy calculation

shows that ϕ is characteristic if and only if the following *resonance condition* is satisfied:

$$(1.5) \qquad (\nabla\phi_\ell - \nabla\phi_m) \cdot (\nabla\phi_\ell - \nabla\phi_k) = 0.$$

Obviously this is a quite severe restriction in one spatial dimension, while in higher dimensions there are many possibilities to satisfy (1.5). In order to gain more insight we shall restrict ourselves from now on to the case of *plane waves* (*i.e.* linear phases, see §2.1), This choice allows for a more detailed mathematical study and is also the most important case from the physical point of view, cf. [31, 13]. The precise mathematical setting is then as follows.

1.3. **Basic mathematical setting and outline.** In the following the space variable $x \in \mathcal{M}$ will either belong to the whole Euclidean space $\mathcal{M} = \mathbb{R}^d$, or to the torus $\mathcal{M} = \mathbb{T}^d$ (we denote $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$), for some $d \in \mathbb{N}$. The latter can be motivated by the fact that numerical simulations of (1.6) are mainly based on pseudo-spectral schemes and thus naturally posed on $\mathbb{T}^d$, see e.g. [2, 3]. We then consider the initial value problem for the slightly more general NLS

$$(1.6) \qquad i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda\varepsilon|u^\varepsilon|^{2\sigma}u^\varepsilon \quad ; \quad u^\varepsilon(0,x) = u_0^\varepsilon(x),$$

where $\sigma \in \mathbb{N}^*$. Although we obtain the most precise results (concerning the geometry of resonances, in particular) in the case of the cubic nonlinearity ($\sigma = 1$), we are in fact able to rigorously justify WKB asymptotics also for higher order nonlinearities. We assume that (1.6) is subject to an initial data u_0^ε, which is assumed to be close (in a sense to be made precise in §6) to superposition of highly oscillatory plane waves, *i.e.*

$$(1.7) \qquad u_0^\varepsilon(x) \approx \sum_{j \in J_0} \alpha_j(x) e^{i\kappa_j \cdot x/\varepsilon},$$

where $J_0 \subseteq \mathbb{Z}$ is a (not necessarily finite) given index set. In the Euclidean case we allow for wave vectors $\kappa_j \in \mathbb{R}^d$, whereas on $\mathcal{M} = \mathbb{T}^d$ we impose $\kappa_j \in \mathbb{Z}^d$. Moreover, in the latter case, we choose α_j to be independent of $x \in \mathbb{T}^d$, so that (1.7) corresponds to an expansion in terms of Fourier series (with $\varepsilon^{-1} \in \mathbb{N}$). The case of x-dependent α_j's on $\mathbb{T}^d$ could be considered as well, by reproducing the analysis on $\mathbb{R}^d$. We choose not to do so here, since it brings no real new information.

In particular, for $x \in \mathbb{T}$ (the one-dimensional torus), our analysis leads to a remarkably simple approximation.

Theorem 1.1. *For $x \in \mathbb{T}$, consider* (1.6) *with $\sigma = 1$. Suppose that the initial data are of the form* (1.7) *with $\kappa_j \equiv j \in \mathbb{Z}$ and $(\alpha_j)_j \in \ell^1(\mathbb{Z})$.*

Then for all $T > 0$, there exist $C = C(T)$ and $\varepsilon_0 > 0$, such that for all $\varepsilon \in]0, \varepsilon_0]$, with $1/\varepsilon \in \mathbb{N}^*$, it holds

$$\sup_{t \in [0,T]} \left\| u^\varepsilon(t) - u_{\mathrm{app}}^\varepsilon(t) \right\|_{L^\infty(\mathbb{T})} \leqslant C\varepsilon,$$

where the approximate solution $u_{\mathrm{app}}^\varepsilon$ is given by

$$u_{\mathrm{app}}^\varepsilon(t,x) = \sum_{j \in \mathbb{Z}} \alpha_j e^{-i\lambda t(2M - |\alpha_j|^2)} e^{i(jx - \frac{1}{2}j^2 t)/\varepsilon}, \ \ and \ M = \sum_{k \in \mathbb{Z}} |\alpha_k|^2.$$

We see that at leading order, the nonlinear interaction shows up through an explicit modulation at scale $\mathcal{O}(1)$. It is well known that the one-dimensional cubic Schrödinger equation is completely integrable (see [18, 24] for the periodic case). However, this aspect does not play any role in the proof of Theorem 1.1, which in itself can be seen as a consequence of the more general result stated in Theorem 6.5. On the other hand, several aspects in the discussion on possible phase resonances and the creation of amplitudes seem to be specific to both properties $d = 1$ and $\sigma = 1$ (see §2 and §3).

In order to prove Theorem 6.5, and henceforth also Theorem 1.1, we need to set up a rigorous multiphase WKB approximation for solutions to (1.6). To this end, there are essentially two steps needed in our analysis. First, we detail the approach sketched above by examining the possible resonances between the phases, and analyzing the evolution and/or the creation of the corresponding profiles a_j. The second step then consists in making this approach rigorous: we construct the profiles a_j, and show that the obtained *ansatz* is a satisfactory approximation of the exact solution u^ε, up to $\mathcal{O}(\varepsilon)$ in a space contained in $L^\infty(\mathcal{M})$. As it is standard, we prove in fact a stronger stability result: Starting from any approximate solution $u^\varepsilon_{\mathrm{app}}$ constructed on profiles, we show that, for any initial data close (as ε goes to zero) to $u^\varepsilon_{\mathrm{app}|t=0}$, there exists an exact solution which is close to $u^\varepsilon_{\mathrm{app}}$, on some time interval independent of ε (which, for ε small enough, may be chosen as any finite time up to which $u^\varepsilon_{\mathrm{app}}$ is defined).

In the case of a single oscillation only, it suffices to multiply u^ε by $e^{-i\phi/\varepsilon}$ to filter out rapid oscillations, see [6]. In the case where several phases are present, this strategy obviously fails. To overcome this issue, a fairly general mathematical approach, which has proved efficient in several contexts (see e.g. [15, 27, 26]), consists in working in rescaled Sobolev spaces, usually denoted by H^s_ε, for $s > 0$. These are the usual Sobolev spaces, where derivatives are scaled by ε, in order to account for the spatial oscillations at scale ε. More precisely, if $s \in \mathbb{N}$,

$$\|f\|^2_{H^s_\varepsilon} := \sum_{|\alpha| \leqslant s} \|(\varepsilon\partial)^\alpha f\|^2_{L^2}.$$

However, due to the negative power of ε in the associated Gagliardo–Nirenberg inequalities, this technique usually demands to construct approximate solution with a high order of precision (see [14] for a closely related study on the interaction of high-frequency waves in periodic potentials). Another, more sophisticated, approach consists in filtering out the rapid oscillations in terms of the free evolution group, as in [29]. In the present work though, we shall use a simpler approach, which allows us to justify the multiphase weakly nonlinear WKB analysis in a remarkably straightforward way. This approach relies on the use of Wiener algebras, as introduced in [20], and further developed in [22, 4, 9]. This analytical framework is particularly convenient in the case of plane waves, but could probably be extended to more general situations, up to some geometric constraints on the phases. However, the first step of the analysis, *i.e.* describing all possible resonances, becomes much more intricate, see e.g. [21, 19].

As well shall see during the course of the proof, the use of Wiener algebras has several advantages on the technical level. We point out that this framework makes it possible to justify the WKB approximation with an error estimate of order $\mathcal{O}(\varepsilon)$

without constructing correctors (which would have to be of order ε or even smaller, when working in H_ε^s spaces, see e.g. [15, 27], or [7, 14] in the NLS case).

1.4. **An application to instability.** As an application of the semi-classical analysis for (1.6), we recover the main result in [8] (see also [5]), concerning NLS in the periodic case. This result has been established in the case $d = 1$, and is hereby extend to higher dimensions. We also propose a variation on a result in [25] (see assertion 3 in the theorem below).

Theorem 1.2. *Let $d \geqslant 1$, $\sigma \in \mathbb{N}^*$ and $\lambda \in \{\pm 1\}$. Fix $s < 0$.*
1. For all $\rho > 0$, we can find a solution u to

$$(1.8) \qquad i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{2\sigma} u, \quad x \in \mathbb{T}^d,$$

with $\|u(0)\|_{H^s(\mathbb{T}^d)} < \rho$, such that for all $\delta > 0$, there exists $\widetilde{u}$ solution to (1.8) with

$$\|u(0) - \widetilde{u}(0)\|_{H^s(\mathbb{T}^d)} < \delta,$$

and

$$\sup_{0 \leqslant t \leqslant \delta} \left| \int_{\mathbb{T}^d} (u(t,x) - \widetilde{u}(t,x))\, dx \right| \geqslant c\rho,$$

for some constant $c > 0$ independent of ρ and δ. In particular, the solution map fails to be continuous as a map from $H^s(\mathbb{T}^d)$ to $H^k(\mathbb{T}^d)$, no matter how close to $-\infty$ the exponent k may be.
2. Suppose $\sigma \geqslant 2$. For any $\rho > 0$ and $\delta > 0$ there exist smooth solutions u, $\widetilde{u}$ of (1.8) such that $u(0) - \widetilde{u}(0)$ is equal to a constant of magnitude at most δ, and

$$\|u(0)\|_{H^s(\mathbb{T}^d)} + \|\widetilde{u}(0)\|_{H^s(\mathbb{T}^d)} \leqslant \rho \quad ; \quad \sup_{0 \leqslant t \leqslant \delta} \left| \int_{\mathbb{T}^d} (u(t,x) - \widetilde{u}(t,x))\, dx \right| \geqslant c\rho,$$

for some constant $c > 0$ independent of ρ and δ.
3. For any $t \neq 0$, the flow-map associated with (1.8) is discontinuous as a map from $L^2(\mathbb{T}^d)$, equipped with its weak topology, into the space of distributions $\left(C^\infty(\mathbb{T}^d)\right)^*$ at any constant $\alpha_0 \in \mathbb{C} \setminus \{0\} \subset L^2(\mathbb{T}^d)$.

We show in §7 that the above instability result can be viewed as a consequence of multiphase weakly nonlinear geometric optics. The first two assertions are an extension of the results in [8], so we shall not comment on their meaning, and refer to the discussion in [8]. We invite the reader to consult [25] for a stronger instability result in the one-dimensional case: indeed, when $d = \sigma = 1$, the author shows the third point in the above statement for any $\alpha_0 \in L^2(\mathbb{T}) \setminus \{0\}$, not necessarily constant.

1.5. **Structure of the paper.** We first study in detail the case of the cubic nonlinearity ($\sigma = 1$). In §2, we consider the set of resonant phases, and in §3, we analyze the corresponding amplitudes. The case of higher order nonlinearities is treated in §4. In §5, we set up the analytical framework, with which a general stability result (of which Theorem 1.1 is a straightforward consequence) is established in §6. Theorem 1.2 is proved in §7. Finally, in an appendix, we sketch how the previous semi-classical analysis can be adapted to more general sets of initial plane waves (including generic finite sets of wave vectors).

2. Analysis of possible resonances in the cubic case

In this section, we show that when $\sigma = 1$, the set of relevant phases can be described in a fairly detailed way.

2.1. General considerations. We seek an approximation of the form

$$u^\varepsilon(t,x) \approx \sum_{j \in J} a_j(t,x) e^{i\phi_j(t,x)/\varepsilon},$$

where here and in the following $J \subset \mathbb{Z}$ denotes the index set of relevant phases ϕ_j (yet to be determined). Note that using J is only a renumbering, so that $j \neq k \Rightarrow \phi_j \neq \phi_k$. In the case $x \in \mathbb{T}^d$, one simply drops the dependence of a_j upon x. In general $J_0 \subsetneq J$, *i.e.* we usually need to take into account more phases in (2.1) than we are given initially.

As a first step we need to determine the characteristic phases $\phi_j(t,x) \in \mathbb{R}$. For plane-wave initial data of the form (1.7) we are led to the following initial value problem

$$\partial_t \phi_j + \frac{1}{2}|\nabla \phi_j|^2 = 0 \quad ; \quad \phi_j(0,x) = \kappa_j \cdot x,$$

the solution of which is explicitly given by

$$(2.1) \qquad\qquad \phi_j(t,x) = \kappa_j \cdot x - \frac{t}{2}|\kappa_j|^2.$$

Recall that for $x \in \mathbb{R}^d$, we assume $\kappa_j \in \mathbb{R}^d$, whereas in the case $x \in \mathbb{T}^d$, we restrict ourselves to $\kappa_j \in \mathbb{Z}^d$. Of course, these phases ϕ_j remain smooth for all time, *i.e.* *no caustic* appears.

In the cubic case $\sigma = 1$, the set of resonances leading to the phase ϕ_j is therefore given by

$$I_j = \{(k,\ell,m) \in J^3 \ ; \ \kappa_k - \kappa_\ell + \kappa_m = \kappa_j, \ |\kappa_k|^2 - |\kappa_\ell|^2 + |\kappa_m|^2 = |\kappa_j|^2\},$$

and the corresponding resonance condition (1.5) becomes

$$(2.2) \qquad\qquad (\kappa_\ell - \kappa_m) \cdot (\kappa_\ell - \kappa_k) = 0.$$

As we shall see, this condition provides several insights on the structure of four-wave resonances.

2.2. The one-dimensional case. For $d = 1$ the condition (2.2) implies that if $(k,\ell,m) \in I_j$, then $\kappa_\ell = \kappa_m$, or $\kappa_\ell = \kappa_k$. Therefore, when $d = 1$, the set I_j is fully described by:

$$I_j = \{(j,\ell,\ell), \ (\ell,\ell,j) \ ; \ \ell \in J\},$$

no new phase can be generated by a cubic interaction.

In higher dimensions, however, the situation is much more complicated and heavily depends on the number of initial modes.

2.3. Multi-dimensional case $d \geqslant 2$. We start with the simplest multiphase situation and proceed from there to more complicated cases. Eventually we shall arrive at a geometric interpretation for the generic case.

2.3.1. *One or two initial modes.* If we start from only two initial modes, $\sharp J_0 = 2$, the resonance condition (2.2) implies that the cubic interaction between these two phases *cannot create* a new characteristic phase. In other words, u^ε exhibits at most two rapid oscillations at leading order. Recalling that $\phi = 0$ is an admissible phase, the case of a single initial phase $\sharp J_0 = 1$, is therefore included (if one of the initial amplitudes is set equal to zero). We want to emphasize that the case of at most two initial phases is rather particular, since (2.2) implies that the situation is the same for all spatial dimensions $d \geqslant 1$.

Remark 2.1. In addition, the fact that two phases cannot create a new one extends also to higher order (gauge invariant) nonlinearities $f(z) = \lambda |z|^{2\sigma} z$, for $\sigma \in \mathbb{N}$, $\sigma \geqslant 2$, see §4.

2.3.2. *Three or four initial modes.* This case can be fully understood by the following geometric insight, already noticed in [10]:

Lemma 2.2. *Let $d \geqslant 2$, and k, ℓ, m belong to J. Then, $(\kappa_k, \kappa_\ell, \kappa_m) \in I_j$ precisely when the endpoints of the vectors $\kappa_k, \kappa_\ell, \kappa_m, \kappa_j$ form* four corners of a non-degenerate rectangle *with κ_ℓ and κ_j opposing each other, or when this quadruplet corresponds to one of the two following degenerate cases: $(\kappa_k = \kappa_j, \kappa_m = \kappa_\ell)$, or $(\kappa_k = \kappa_\ell, \kappa_m = \kappa_j)$.*

Remark 2.3. In the degenerate cases, no new phase is created.

Proof. We recall the argument given in [10], by first noting that the relations between $(\kappa_j, \kappa_k, \kappa_\ell, \kappa_m)$ formulated in (2.1), are equivalently fulfilled by $(\kappa_k - \kappa, \kappa_\ell - \kappa, \kappa_m - \kappa, \kappa_j - \kappa)$, for any $\kappa \in \mathbb{R}^d$ (resp. $\kappa \in \mathbb{Z}^d$). This is easily seen by expanding the second relation in (2.1) and inserting the first one. Thus, choosing $\kappa = \kappa_j$, it therefore suffices to prove this geometric interpretation for $\kappa_j = 0$, which consequently shows: $\kappa_k + \kappa_m = \kappa_\ell$ such that $\kappa_k \cdot \kappa_m = 0$, by the law of cosines. $\square$

In summary, we conclude that three initial (plane-wave) phases *create at most one new phase*, such that the corresponding four wave vectors form a rectangle. When the initial wave vectors $\{\kappa_j\}_{j \in J_0}$ are chosen such that their endpoints form the four corners of a rectangle, no new phase can be created by the cubic nonlinearity and u^ε exhibits only four rapid oscillations. We close this subsection with two illustrative examples.

Example 2.4. Let $d = 2$. Consider $\kappa_1 = (0, 1)$, $\kappa_2 = (1, 1)$ and $\kappa_3 = (1, 0)$. The cubic interaction creates the zero mode $\phi_4 \equiv 0$.

Example 2.5. Again let $d = 2$, with now $\kappa_1 = (1, 1)$, $\kappa_2 = (1, 2)$ and $\kappa_3 = (3, 2)$. In this case, we create a non-zero phase ϕ_4, with corresponding wave vector $\kappa_4 = (3, 1)$.

The geometric insight gained above then directly leads us to the following description of the resonant set I_j in the general case.

2.3.3. *The general case.* We are given a countable (possibly finite) number of initial phases $\{\phi_j\}_{j \in J_0}$ with corresponding wave vectors $\{\kappa_j\}_{j \in J_0}$. From the discussion of the previous paragraph it is clear that there are two possible situations:

  (a) Either, it is *impossible* to create a *new* rectangle from any possible subset $\tilde{J}_0 \subset J_0$, such that $\sharp \tilde{J}_0 = 3$. If so, then no new phase can be created. This is the generic case.

(b) Or, starting from an initial (finite or countable) set $S_0 = \{\kappa_j\}_{j \in J_0}$, we may obtain a first generation $S_1 = \{\kappa_j\}_{j \in J_1}$ with $J_0 \subset J_1$ (*i.e.* $S_0 \subset S_1$) in the following way: we add to S_0 all points $\kappa \in \mathbb{R}^d$, such that there exist $\tilde{J}_0 \subset J_0$ with $\sharp \tilde{J}_0 = 3$, and such that $\{\kappa_j\}_{\tilde{J}_0} \cup \{\kappa\}$ is a rectangle. Note that, if $J_0 \subset \mathbb{Z}^d$, then $J_1 \subset \mathbb{Z}^d$. By a recursive scheme, we are led to a (finite or countable) set S which is stable under the completion of right-angled triangles formed of points from this set, into rectangles. Furthermore, if $S_0 \subset \mathbb{Z}^d$, then $S \subset \mathbb{Z}^d$.

Example 2.6. As already seen, the simplest examples for possibility (a) are the cases $\sharp J_0 \leqslant 3$ when the triangle formed by the endpoints of the considered wave vectors has no right angle, or $\sharp J_0 = 4$, where the four initial phases are chosen such that their corresponding wave vectors $\{\kappa_j\}_{j \in J_0}$ already form the corners of a rectangle.

From a finite number of initial phases, possibility (b) may lead to a finite as well as to an infinite set J. Even for $d = 2$, we have:

Example 2.7. In the plane $\mathbb{R}^2$, start with

$$J_0 = \{(-1,1), (0,1), (0,0), (1,0)\}.$$

The first generation is then

$$J_1 = \{(-1,1), (0,1), (1,1), (-1,0), (0,0), (1,0)\} = J_0 \cup \{(1,1), (-1,0)\},$$

and the second one is

$$J_2 = J_1 \cup \{(0,2), (0,-1)\}.$$

One easily sees that this generates $J = \mathbb{Z}^2$.

As a conclusion, the set of phases $\{\phi_j\}_{j \in J}$ may be finite or infinite, but has the following property.

Proposition 2.8. *Let $\sigma = 1$, and consider any triplet of wave vectors from $S = \{\kappa_j\}_{j \in J}$. Then, either the corresponding triangle has no right angle, or the fourth corner of the associated rectangle belongs to S.*

3. Analysis of the amplitude system in the cubic case

From the previous section, in general we have to expect the generation of new phases by the four-wave resonance. However, it may happen that not all of them are actually present in our approximation (2.1), since the corresponding profile $a_j(t, x)$ has to be non-trivial.

Indeed, if we plug the ansatz (2.1) into (1.6) the terms of order $\mathcal{O}(1)$ are identically zero since all the ϕ_j's are characteristic. For the $\mathcal{O}(\varepsilon)$ term, we project on the oscillations associated to ϕ_j, which yields the following system of transport equations:

$$(3.1) \quad \forall j \in J, \quad \partial_t a_j + \kappa_j \cdot \nabla a_j = -i\lambda \sum_{(k,\ell,m) \in I_j} a_k \overline{a}_\ell a_m \quad ; \quad a_j(0, x) = \alpha_j(x),$$

with obviously $\nabla a_j = 0$ in the case where $x \in \mathbb{T}^d$. In the following we will perform a qualitative analysis of the system (3.1), postponing the rigorous existence and uniqueness analysis to §5.4. Having in mind the discussion from §2 we distinguish the case $d = 1$ from the case $d \geqslant 2$.

3.1. **The case $d = 1$.** Let $j \in J$, and recall that I_j is particularly simple in $d = 1$:

$$I_j = \{(j, \ell, \ell), \ (\ell, \ell, j) \ ; \ \ell \in J\}.$$

Using this, (3.1) simplifies to

$$(3.2) \qquad (\partial_t + \kappa_j \partial_x) \, a_j = -2i\lambda \sum_{\ell \in J} |a_\ell|^2 a_j + i\lambda |a_j|^2 a_j \quad ; \quad a_j(0, x) = \alpha_j(x).$$

In particular, the evolution of a zero profile $\alpha_j \equiv 0$ is necessarily trivial, that is $a_j(t, x) \equiv 0$. This non-generation of profiles leads to the same conclusion as §2.2: No new mode can be created, if it is not present initially (and the reason is the same as in §2.2: a_j factors out in (3.2) just because for any $(\ell_1, \ell_2, \ell_3) \in I_j$, we have $\ell_1 = j$ or $\ell_3 = j$). We shall see that the multi-dimensional situation is quite different but first examine the situation for $x \in \mathbb{T}$ and $x \in \mathbb{R}$ in more detail.

3.1.1. *The case $x \in \mathbb{T}$.* In this case, we readily obtain that $|a_j|^2$ does not depend on time. This is due to the fact that (3.2) yields: $i\partial_t a_j \in \mathbb{R} a_j$ and hence $\partial_t |a_j|^2 = 0$, for all $j \in \mathbb{Z}$. In particular we get that

$$M = \|u^\varepsilon(0)\|_{L^2}^2 = \sum_{j \in J} |\alpha_j|^2 = \|u^\varepsilon(t)\|_{L^2}^2, \quad \forall t \in \mathbb{R}.$$

The conserved quantity M corresponds to the total mass of the exact solution u^ε. Using this, we rewrite (3.2) as

$$\frac{d}{dt} a_j = -i\lambda \left(2M - |\alpha_j|^2\right) a_j,$$

which yields an explicit formula for the (global in time) solution

$$a_j(t) = \alpha_j e^{-i\lambda t \left(2M - |\alpha_j|^2\right)}.$$

We observe that in the case of the one-dimensional torus, the interaction of the profiles a_j is particularly simple. Nonlinear effects lead to phase-modulations only.

3.1.2. *The case $x \in \mathbb{R}$.* Here, in contrast to the situation on $\mathbb{T}$, the modulus of a_j is no longer conserved, since we can only conclude from (3.2) that

$$(\partial_t + \kappa_j \partial_x) \, |a_j|^2 = 0,$$

and thus

$$|a_j(t, x)|^2 = |\alpha_j(x - t\kappa_j)|^2.$$

In particular we readily see that for all $j \in J$ we have

$$(3.3) \qquad \|a_j(t)\|_{L^2} = \|\alpha_j\|_{L^2}, \quad \forall t \in \mathbb{R}.$$

Moreover, we still have an explicit representation for the solution of (3.2) in the form

$$(3.4) \qquad a_j(t, x) = \alpha_j(x - t\kappa_j) e^{iS_j(t, x)},$$

for some real-valued phase S_j, yet to be computed. In view of the identity

$$(\partial_t + \kappa_j \partial_x) \, a_j(t, x) = i\alpha_j(x - t\kappa_j) e^{i\lambda S_j(t, x)} (\partial_t + \kappa_j \partial_x) \, S_j(t, x),$$

equation (3.2) implies

$$((\partial_t + \kappa_j \partial_x) S_j(t,x)) \alpha_j(x - t\kappa_j) = \lambda \left(-2 \sum_{\ell \in J} |\alpha_\ell(x - t\kappa_\ell)|^2 + |\alpha_j(x - t\kappa_j)|^2 \right)$$
$$\times \alpha_j(x - t\kappa_j).$$

One easily sees that it is sufficient to impose

$$\partial_t (S_j(t, x + t\kappa_j)) = -2\lambda \sum_{\ell \in J} |\alpha_\ell(x + t(\kappa_j - \kappa_\ell))|^2 + \lambda |\alpha_j(x)|^2,$$

which yields

$$(3.5) \qquad S_j(t,x) = -2\lambda \int_0^t \left(\sum_{\ell \in J \setminus \{j\}} |\alpha_\ell(x + (\tau - t)\kappa_j - \tau\kappa_\ell))|^2 d\tau \right)$$
$$- t\lambda |\alpha_j(x - t\kappa_j)|^2.$$

This formula, together with (3.4) describes the modulation of the profile $a_j(t,x)$. As in the case of the torus, amplitudes are transported linearly. Only the (slow) phases S_j undergo nonlinear effects, which are more complicated as before but still explicitly described in terms of the initial data.

3.2. **The case of one or two modes for $d \geqslant 1$.** We have already seen in §2.3.1 that the case of two initial modes is special, since we get a closed system for all $d \geqslant 1$. Indeed if we start from two phases and two associated profiles, say a_j and a_ℓ, the system (3.1) simplifies to:

$$\partial_t a_j + \kappa_j \cdot \nabla a_j = -i\lambda \left(|a_j|^2 + 2|a_\ell|^2 \right) a_j, \quad ; \quad a_j(0,x) = \alpha_j(x),$$
$$\partial_t a_\ell + \kappa_\ell \cdot \nabla a_\ell = -i\lambda \left(2|a_j|^2 + |a_\ell|^2 \right) a_\ell, \quad ; \quad a_\ell(0,x) = \alpha_\ell(x).$$

Note that if initially one of the two profiles is identically zero, it remains zero for all times and hence, we are back in the situation of a usual single-phase WKB approximation. In particular we compute explicitly for:

- Two modes, on $\mathbb{T}^d$:

$$a_j(t) = \alpha_j e^{-i\lambda t(2|\alpha_\ell|^2 + |\alpha_j|^2)} \quad ; \quad a_\ell(t) = \alpha_\ell e^{-i\lambda t(2|\alpha_j|^2 + |\alpha_\ell|^2)}.$$

- Two modes, on $\mathbb{R}^d$:

$$a_j(t,x) = \alpha_j(x - t\kappa_j) e^{-i\lambda \left(2 \int_0^t |\alpha_\ell(x + (\tau - t)\kappa_j - \tau\kappa_\ell)|^2 d\tau + t|\alpha_j(x - t\kappa_j)|^2 \right)},$$
$$a_\ell(t,x) = \alpha_\ell(x - t\kappa_\ell) e^{-i\lambda \left(2 \int_0^t |\alpha_j(x + (\tau - t)\kappa_\ell - \tau\kappa_j)|^2 d\tau + t|\alpha_\ell(x - t\kappa_\ell)|^2 \right)}.$$

Again, these solutions exhibit (nonlinear) self-modulation of phases only, and exist for all times $t \in \mathbb{R}$, a property which is a-priori not clear in the general case.

3.3. **Creation of new modes when $d \geqslant 2$.** A basic difference between the one-dimensional case and the multidimensional situation is that the conservation law (3.3) does not remain valid when $d \geqslant 2$. However, we are still able to prove that the total mass is conserved.

Lemma 3.1. *For any solution to* (3.1) *it holds*

$$(3.6) \qquad \frac{d}{dt} \sum_{j \in J} \|a_j(t)\|_{L^2}^2 = 0.$$

Proof. The assertion follows from the more general identity

$$\sum_{j \in J} \left(\partial_t + \kappa_j \cdot \nabla \right) |a_j|^2 = 0,$$

since, by definition we have

$$\sum_{j \in J} \left(\partial_t + \kappa_j \cdot \nabla \right) |a_j|^2 = \operatorname{Im} \left(\lambda \sum_{j \in J} \sum_{(k,\ell,m) \in I_j} \overline{a}_j a_k \overline{a}_\ell a_m \right).$$

This sum is zero by symmetry, since for each quadruplet $(j, k, \ell, m) \in J^4$ of indices the quadruplet (j, m, ℓ, k) is also present, as well as the other six obtained by circular permutation (at least in the nondegenerate case mentioned in Lemma 2.2; adaptation to the degenerate case is obvious). These are the only occurrences of the corresponding rectangle of wave numbers, and they produce the sum

$$2 \left(\overline{a}_j a_k \overline{a}_\ell a_m + \overline{a}_k a_\ell \overline{a}_m a_j + \overline{a}_\ell a_m \overline{a}_j a_k + \overline{a}_m a_j \overline{a}_k a_\ell \right) = 8 \operatorname{Re} \left(\overline{a}_j a_k \overline{a}_\ell a_m \right),$$

which is real. We consequently infer

$$\partial_t \sum_{j \in J} |a_j(t, x + t\kappa_j)|^2 = 0,$$

and thus also

$$\frac{d}{dt} \sum_{j \in J} \|a_j(t, \cdot + t\kappa_j)\|_{L^2}^2 = \frac{d}{dt} \sum_{j \in J} \|a_j(t, \cdot)\|_{L^2}^2 = 0.$$

$\square$

Let us now turn to the possibility of creating new profiles by nonlinear interactions (note however that the conservation law (3.6) gives a global constraint on this process). To simplify the presentation, we assume $d = 2$. The creation of new oscillations in the general case $d \geqslant 2$ then follows by completing elements in $\mathbb{R}^2$ with $(0, \dots, 0) \in \mathbb{R}^{d-2}$ and analogously for the situation on $\mathbb{T}^d$. Consider the geometry associated to Example 2.4: We thus have (on $\mathbb{T}^d$ or $\mathbb{R}^d$)

$$i\partial_t a_0 = \lambda \sum_{(k,\ell,m) \in I_0} a_k \overline{a}_\ell a_m.$$

Recall that $(k, \ell, m) \in I_0$ if and only if

$$\kappa_k - \kappa_\ell + \kappa_m = 0 \quad ; \quad |\kappa_k|^2 - |\kappa_\ell|^2 + |\kappa_m|^2 = 0,$$

which obviously implies $\kappa_k \cdot \kappa_m = 0$. Such a possibility occurs in two cases:

- $\kappa_k = 0$ or $\kappa_m = 0$.
- $(\kappa_k, \kappa_m) = (\kappa_1, \kappa_3)$ or $(\kappa_k, \kappa_m) = (\kappa_3, \kappa_1)$ and hence $\kappa_\ell = \kappa_2$.

From these various cases, we infer

$$i\partial_t a_0 = \lambda \left(|a_0|^2 + 2|a_1|^2 + |a_2|^2 + 2|a_3|^2 \right) a_0 + 2a_1 \overline{a}_2 a_3.$$

Consider three non-vanishing initial oscillations, such that $a_1 \overline{a}_2 a_{3|t=0} \neq 0$. Thus, even if $a_{0|t=0} = 0$, we have $\partial_t a_{0|t=0} \neq 0$, and this (non-oscillating) fourth mode is instantaneously non-vanishing.

4. Higher order nonlinearities

4.1. **Analysis of possible resonances.** So far we were only concerned with four-wave interactions corresponding to cubic nonlinearities, *i.e.* $\sigma = 1$ in (1.6). In general though, the set of resonances associated to a (gauge invariant) nonlinearity of the form $f(z) = \lambda|z|^{2\sigma}z$, $\sigma \in \mathbb{N}$, are defined by

$$I_j^\sigma = \Big\{ (\ell_1, \ldots, \ell_{2\sigma+1}) \in J^{2\sigma+1}; \ \sum_{k=1}^{2\sigma+1} (-1)^{k+1} \kappa_{\ell_k} = \kappa_j, \ \sum_{k=1}^{2\sigma+1} (-1)^{k+1} |\kappa_{\ell_k}|^2 = |\kappa_j|^2 \Big\}.$$

As in Section 2, the set of wave vectors $\{\kappa_j\}_{j \in J}$ is constructed by induction, starting from an a finite or countable set $\{\kappa_j\}_{j \in J_0}$, to which we first add a vector κ when there exist $\kappa_{\ell_1}, \ldots, \kappa_{\ell_{2\sigma+1}} \in J_0$ such that

$$(4.1) \qquad \sum_{k=1}^{2\sigma+1} (-1)^{k+1} |\kappa_{\ell_k}|^2 = \left| \sum_{k=1}^{2\sigma+1} (-1)^{k+1} \kappa_{\ell_k} \right|^2;$$

we then set $\kappa = \sum_{k=1}^{2\sigma+1} (-1)^{k+1} \kappa_{\ell_k}$. The same iterative procedure as in §2.3.3 leads to the following analogue to Proposition 2.8:

Proposition 4.1. *Let $\sigma \geqslant 2$, and consider any $(2\sigma + 1)$-tuple $(\kappa_{\ell_1}, \ldots, \kappa_{\ell_{2\sigma+1}})$ of wave vectors from $S = \{\kappa_j\}_{j \in J}$. Then, either the relation (4.1) is not satisfied, or the vector $\kappa_j = \sum_{k=1}^{2\sigma+1} (-1)^{k+1} \kappa_{\ell_k}$ belongs to S.*

Remark 4.2. It is worth noting that, even if we only have very poor information on the set of wave vectors $\{\kappa_j\}_{j \in J}$, it is however a subset of the group generated by the initial set $\{\kappa_j\}_{j \in J_0}$.

The profile equations, analogue to (3.1), are then, for all $j \in J$:

$$(4.2) \quad \partial_t a_j + \kappa_j \cdot \nabla a_j = -i\lambda \sum_{(\ell_1, \ldots, \ell_{2\sigma+1}) \in I_j} a_{\ell_1} \overline{a}_{\ell_2} \ldots a_{\ell_{2\sigma+1}} \quad ; \quad a_j(0, x) = \alpha_j(x).$$

4.2. **The case of two modes.** Similar to the situation for $\sigma = 1$, the case of only two initial modes is rather special. Indeed, the fact that two phases cannot create a new one extends also to higher order nonlinearities. In order to explain the argument, consider first a quintic nonlinearity, corresponding to $\sigma = 2$. To obtain a nonlinear resonance, the wave vectors need to satisfy

$$\kappa_k - \kappa_\ell + \kappa_m - \kappa_p + \kappa_q = \kappa_j,$$

$$|\kappa_k|^2 - |\kappa_\ell|^2 + |\kappa_m|^2 - |\kappa_p|^2 + |\kappa_q|^2 = |\kappa_j|^2,$$

where $k, \ell, m, p, q \in \{j_1, j_2\}$, $j_1, j_2 \in J$. First, if j_1 (or j_2) appears at least twice on the left hand side, with at least one plus and one minus, then the cancellation reduces the discussion to the one we had about the cubic nonlinearity. Hence, no new resonant phase can be created in this case. The complementary case corresponds, up to exchanging j_1 and j_2, to

$$\kappa_k = \kappa_m = \kappa_q = \kappa_{j_1} \quad \text{and} \quad \kappa_\ell = \kappa_p = \kappa_{j_2}.$$

The above relations yield

$$3\kappa_{j_1} - 2\kappa_{j_2} = \kappa_j \quad ; \quad 3|\kappa_{j_1}|^2 - 2|\kappa_{j_2}|^2 = |\kappa_j|^2.$$

Squaring the first identity and comparing with the second one, we infer

$$6|\kappa_{j_1} - \kappa_{j_2}|^2 = 0.$$

Therefore, no new resonant phase can be created by the quintic interaction of two initial resonant plane waves.

Consider now the general case where $\sigma \geqslant 2$: The same argument as above shows that the only new case is the one where all the plus signs correspond to one phase, and all the minus signs to the other:

$$(\sigma + 1)\kappa_{j_1} - \sigma\kappa_{j_2} = \kappa_j \quad ; \quad (\sigma + 1)|\kappa_{j_1}|^2 - \sigma|\kappa_{j_2}|^2 = |\kappa_j|^2.$$

Squaring the first identity and comparing with the second one, we infer

$$\sigma(\sigma + 1)|\kappa_{j_1} - \kappa_{j_2}|^2 = 0.$$

We conclude as above, and obtain the following result:

Proposition 4.3. *Let $\sigma \in \mathbb{N}^*$, and let $\kappa_1, \kappa_2 \in \mathbb{R}^d$ be such that $\kappa_1 \neq \kappa_2$. To these wave vectors are associated the characteristic phases*

$$\phi_j(t, x) = \kappa_j \cdot x - \frac{t}{2}|\kappa_j|^2, \quad j = 1, 2.$$

Then, these two phases can not create no new phase by $(2\sigma+1)$th-order interaction: the set

$$\left\{ \kappa \in \mathbb{R}^d \mid \exists (\ell_1, \ldots, \ell_{2\sigma+1}) \in \{1, 2\}^{2\sigma+1}, \quad \kappa = \sum_{k=1}^{2\sigma+1} (-1)^{k+1} \kappa_{\ell_k} \right.$$

$$\left. and \quad |\kappa|^2 = \sum_{k=1}^{2\sigma+1} (-1)^{k+1}|\kappa_{\ell_k}|^2 \right\}$$

is reduced to $\{\kappa_1, \kappa_2\}$.

In view of Proposition 4.3, the system (4.2) becomes a system of two equations, which can be integrated explicitly, as in [8, Remark 3.1]:

$$(4.3) \quad \begin{aligned} &\partial_t a_j + \kappa_j \cdot \nabla a_j = -i\lambda \sum_{n=0}^{\sigma} \binom{\sigma + 1}{n} \binom{\sigma}{n} |a_j|^{2\sigma - 2n} |a_\ell|^{2n} a_j, \\ &\partial_t a_\ell + \kappa_\ell \cdot \nabla a_\ell = -i\lambda \sum_{n=0}^{\sigma} \binom{\sigma + 1}{n} \binom{\sigma}{n} |a_\ell|^{2\sigma - 2n} |a_j|^{2n} a_\ell. \end{aligned}$$

In the case of $\mathbb{T}^d$, we find for instance

$$(4.4) \quad \begin{aligned} &a_j(t) = \alpha_j \exp\left(-i\lambda t \sum_{n=0}^{\sigma} \binom{\sigma + 1}{n} \binom{\sigma}{n} |\alpha_j|^{2\sigma - 2n} |\alpha_\ell|^{2\ell} \right), \\ &a_\ell(t) = \alpha_\ell \exp\left(-i\lambda t \sum_{n=0}^{\sigma} \binom{\sigma + 1}{n} \binom{\sigma}{n} |\alpha_\ell|^{2\sigma - 2n} |\alpha_j|^{2n} \right). \end{aligned}$$

In the case of $\mathbb{R}^d$, the formula is more intricate and we shall omit it.

Apart from the two-phase situation, the results for of Section 2.3.3 on resonances do not carry over to the general case $\sigma \geqslant 2$ in any straightforward manner. Even in space dimension $d = 1$, the resonant sets cease to be as simple for $\sigma \geqslant 2$, provided that one starts with at least three modes.

Example 4.4. Consider the quintic case $\sigma = 2$ in $d = 2$ spatial dimensions. As we have seen above a resonance for such a quintic nonlinearity appears if and only if

$$\kappa_k - \kappa_\ell + \kappa_m - \kappa_p + \kappa_q = \kappa_j,$$
$$|\kappa_k|^2 - |\kappa_\ell|^2 + |\kappa_m|^2 - |\kappa_p|^2 + |\kappa_q|^2 = |\kappa_j|^2.$$

We can pick for instance three initial phases of the form

$$\kappa_1 = (-1, 0) \quad ; \quad \kappa_2 = (0, 0) \quad ; \quad \kappa_3 = (2, 0).$$

For $k = 1$, $\ell = p = 2$, $m = q = 3$, we have a resonance, creating $\kappa_4 = (3, 0)$, whereas in the case $\sigma = 1$, no resonance occurs between the phases with wave vectors κ_1, κ_2 and κ_3. This example shows that the geometric characterization of four-wave resonances given in §2.3.2 does not export to the case of six-wave resonances: κ_1, κ_2, κ_3 and κ_4 all belong to the line $x_2 = 0$.

Example 4.5. Consider the same example as above in $d = 1$. *i.e.* pick three initial phases of the form

$$\kappa_1 = -1 \quad ; \quad \kappa_2 = 0 \quad ; \quad \kappa_3 = 2$$

and create a resonance $\kappa_4 = 3$ for $k = 1$, $\ell = p = 2$, $m = q = 3$. This is in sharp contrast to the case $\sigma = 1$, where no new phases can be created in $d = 1$. Moreover, a non-vanishing amplitude a_4 is effectively generated:

$$(\partial_t + \kappa_4 \partial_x) a_4 = - 3i\lambda \left(|a_1|^4 + |a_3|^4 + |a_3|^4 + 4(|a_1|^2 |a_2|^2 + |a_2|^2 |a_3|^2 + |a_3|^2 |a_1|^2) \right) a_4$$
$$- i6\lambda a_1 \overline{a_2}^2 a_3^2 - 6i\lambda \overline{a}_1 |a_2|^2 a_3.$$

We see that we may have $a_{4|t=0} = 0$, but

$$(\partial_t a_4)_{|t=0} = \left(-i6\lambda a_1 \overline{a_2}^2 a_3^2 - 6i\lambda \overline{a}_1 |a_2|^2 a_3 \right)_{|t=0} \neq 0,$$

showing the appearance of a non-trivial a_4 for $t > 0$.

Despite this lack of knowledge concerning the precise structure of possible resonances for higher order nonlinearities, we shall see that we are able to prove the validity of WKB approximation even in this case.

5. ANALYTICAL FRAMEWORK

We now present the analytical framework needed for the rigorous justification of a multiphase WKB approximation.

5.1. **Wiener algebras.** On $\mathcal{M} = \mathbb{T}^d$, we consider the usual Wiener algebra of functions with absolutely summable Fourier series:

Definition 5.1 (Wiener algebra on $\mathcal{M} = \mathbb{T}^d$). *Functions of the form*

$$f(y) = \sum_{k \in \mathbb{Z}} b_k e^{i\kappa_k \cdot y}, \quad \text{with } \kappa_k \in \mathbb{Z}^d \text{ and } b_k \in \mathbb{C},$$

belong to $W(\mathbb{T}^d)$ if and only if $(b_k)_{k \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$. We denote

$$\|f\|_W = \sum_{k \in \mathbb{Z}} |b_k|.$$

In the sequel, when $x \in \mathbb{T}^d$, we consider initial data for (1.6) which are of the form $f(x/\varepsilon)$, with $f \in W(\mathbb{T}^d)$ and $\varepsilon^{-1} \in \mathbb{N}^*$.

Lemma 5.2. *Let f belong to $W(\mathbb{T}^d)$. Then, for all $\varepsilon > 0$ such that $\varepsilon^{-1} \in \mathbb{N}^*$, we have $f(\cdot/\varepsilon) \in W(\mathbb{T}^d)$, and*

$$\|f(\cdot/\varepsilon)\|_W = \|f\|_W.$$

For $\mathcal{M} = \mathbb{R}^d$, the framework is a bit different. Define the Fourier transform by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx.$$

With this normalization, we have $\mathcal{F}^{-1}f(x) = \mathcal{F}f(-x)$. Following [20] and [9], we use on $\mathbb{R}^d$ two different Wiener-type algebras: For the exact solution we use $W(\mathbb{R}^d)$, *i.e.* the space of functions with Fourier transform in $L^1(\mathbb{R}^d)$, and for the profiles, we use $\mathbf{A}(\mathbb{R}^d)$, the space of almost periodic $W(\mathbb{R}^d)$-valued functions on $\mathbb{R}^d$, with absolutely summable Fourier series. We also set $\mathbf{A}(\mathbb{T}^d) = W(\mathbb{T}^d)$, equipped with the same norm.

Definition 5.3 (Wiener algebra on $\mathcal{M} = \mathbb{R}^d$)**.** *We define*

$$W(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d; \mathbb{C}), \quad \|f\|_W := \|\widehat{f}\|_{L^1(\mathbb{R}^d)} < \infty \right\}.$$

Functions of the form

$$f(x,y) = \sum_{k\in\mathbb{Z}} b_k(x) e^{i\kappa_k \cdot y}, \quad \text{with } \kappa_k \in \mathbb{R}^d \text{ and } b_k \in W(\mathbb{R}^d),$$

belong to $\mathbf{A}(\mathbb{R}^d)$ if and only if

$$\|f\|_{\mathbf{A}} := \sum_{k\in\mathbb{Z}} \|b_k\|_W = \sum_{k\in\mathbb{Z}} \|\widehat{b_k}\|_{L^1(\mathbb{R}^d)} < \infty.$$

In the sequel, when $x \in \mathbb{R}^d$, we consider initial data for (1.6) which are of the form $f(x, x/\varepsilon)$, with $f \in \mathbf{A}(\mathbb{R}^d)$. Again, we have

Lemma 5.4. *Let $f \in \mathbf{A}(\mathbb{R}^d)$ and $\varepsilon > 0$. Then $f(\cdot, \cdot/\varepsilon) \in W(\mathbb{R}^d)$ and*

$$\|f(\cdot, \cdot/\varepsilon)\|_W \leqslant \|f\|_{\mathbf{A}}.$$

Proof. We simply have, when $f(x,y) = \sum_{k\in\mathbb{Z}} b_k(x) e^{i\kappa_k \cdot y}$:

$$\|f(\cdot, \cdot/\varepsilon)\|_W = \Big\| \sum_{k\in\mathbb{Z}} \widehat{b_k}(\cdot - \kappa_k/\varepsilon) \Big\|_{L^1(\mathbb{R}^d)} \leqslant \sum_{k\in\mathbb{Z}} \|\widehat{b_k}(\cdot - \kappa_k/\varepsilon)\|_{L^1(\mathbb{R}^d)} = \sum_{k\in\mathbb{Z}} \|\widehat{b_k}\|_{L^1(\mathbb{R}^d)}.$$

The last term is, by definition, $\|f\|_{\mathbf{A}}$. $\qquad\square$

Denote (in the periodic setting as well as in the Euclidean case)

$$U^\varepsilon(t) = e^{i\varepsilon\frac{t}{2}\Delta}.$$

The following properties will be useful (see [9], and also [20, 22, 4]).

Lemma 5.5. *Let $\mathcal{M} = \mathbb{T}^d$ or $\mathbb{R}^d$.*
1. $W(\mathcal{M})$ is a Banach space, continuously embedded into $L^\infty(\mathcal{M})$.
2. $W(\mathcal{M})$ is an algebra, in the sense that the mapping $(f,g) \mapsto fg$ is continuous from $W(\mathcal{M})^2$ to $W(\mathcal{M})$, and moreover

$$\forall f, g \in W(\mathcal{M}), \quad \|fg\|_W \leqslant \|f\|_W \|g\|_W.$$

3. If $F : \mathbb{C} \to \mathbb{C}$ maps u to a finite sum of terms of the form $u^p \overline{u}^q$, $p, q \in \mathbb{N}$, then it extends to a map from $W(\mathcal{M})$ to itself which is uniformly Lipschitzean on bounded sets of $W(\mathcal{M})$.
4. For all $t \in \mathbb{R}$, $U^\varepsilon(t)$ is unitary on $W(\mathcal{M})$.

5.2. **Action of the free Schrödinger group on** $W(\mathcal{M})$. As it is standard for solutions to the equation

$$i\varepsilon \partial_t w^\varepsilon + \frac{\varepsilon^2}{2}\Delta w^\varepsilon = F^\varepsilon,$$

we will consider the corresponding Duhamel's formula

$$w^\varepsilon(t,x) = U^\varepsilon(t)w^\varepsilon(0,x) - i\varepsilon^{-1}\int_0^t U^\varepsilon(t-\tau)F^\varepsilon(\tau,x)d\tau.$$

In view of this representation formula we first need to study the action of the free Schrödinger group $U^\varepsilon(t)$ on $W(\mathcal{M})$.

5.2.1. *The case* $\mathcal{M} = \mathbb{T}^d$. The action of $U^\varepsilon(t)$ on Fourier series on $\mathbb{T}^d$ is well understood. For $\sum_{k\in\mathbb{Z}} b_k e^{i\kappa_k \cdot y} \in W(\mathbb{T}^d)$:

$$(5.1) \qquad U^\varepsilon(t)\left(\sum_{k\in\mathbb{Z}} b_k e^{i\kappa_k \cdot x/\varepsilon}\right) = \sum_{k\in\mathbb{Z}} b_k e^{i\kappa_k \cdot x/\varepsilon - i|\kappa_k|^2 t/(2\varepsilon)}.$$

In view of Duhamel's formula, we will use the following

Lemma 5.6. *Let* $T > 0$, $\omega \in \mathbb{Z}$, $\kappa \in \mathbb{Z}^d$, *and* $b, \partial_t b \in L^\infty([0,T])$. *Denote*

$$D^\varepsilon(t,x) := \int_0^t U^\varepsilon(t-\tau)\left(b(\tau)e^{i\kappa\cdot x/\varepsilon - i\omega\tau/(2\varepsilon)}\right)d\tau.$$

1. *We have* $D^\varepsilon \in C([0,T]\times\mathbb{T}^d)$ *and*

$$\|D^\varepsilon\|_{L^\infty([0,T]\times\mathbb{T}^d)} \leqslant \int_0^T |b(t)|dt.$$

2. *Assume* $\omega \neq |\kappa|^2$. *Then there exists* C *independent of* κ, ω *and* b *such that*

$$\|D^\varepsilon\|_{L^\infty([0,T]\times\mathbb{T}^d)} \leqslant \frac{C\varepsilon}{||\kappa|^2 - \omega|}\left(\|b\|_{L^\infty([0,T])} + \|\partial_t b\|_{L^\infty([0,T])}\right).$$

Proof. In view of the identity (5.1), we have

$$D^\varepsilon(t,x) = \int_0^t b(\tau)e^{i\kappa\cdot x/\varepsilon - i\omega\tau/(2\varepsilon)}e^{-i|\kappa|^2(t-\tau)/(2\varepsilon)}d\tau$$

$$= e^{i\kappa\cdot x/\varepsilon - i|\kappa|^2 t/(2\varepsilon)}\int_0^t b(\tau)e^{i(|\kappa|^2 - \omega)\tau/2\varepsilon}d\tau.$$

The first point is straightforward. Integration by parts yields, since by assumption $|\kappa|^2 - \omega \in \mathbb{Z}\setminus\{0\}$: with $\phi(t,x) = \kappa\cdot x - |\kappa|^2 t/2$,

$$D^\varepsilon(t,x) = e^{i\phi(t,x)/\varepsilon}\left(-\frac{2\varepsilon i}{|\kappa|^2 - \omega}b(\tau)e^{i(|\kappa|^2 - \omega)\tau/2\varepsilon}\Big|_0^t\right.$$

$$\left. + \frac{2\varepsilon i}{|\kappa|^2 - \omega}\int_0^t \partial_t b(\tau)e^{i(|\kappa|^2 - \omega)\tau/2\varepsilon}d\tau\right).$$

The lemma then follows easily. $\qquad\square$

5.2.2. *The case $\mathcal{M} = \mathbb{R}^d$.* The Euclidean counterpart of Lemma 5.6 is a little bit more delicate:

Lemma 5.7. *Let $T > 0$, $\omega \in \mathbb{R}$, $\kappa \in \mathbb{R}^d$, and $b \in L^\infty([0, T]; W(\mathbb{R}^d))$. Denote*

$$D^\varepsilon(t, x) := \int_0^t U^\varepsilon(t - \tau) \left(b(\tau, x) e^{i\kappa \cdot x/\varepsilon - i\omega\tau/(2\varepsilon)} \right) d\tau.$$

1. We have $D^\varepsilon \in C([0, T]; W(\mathbb{R}^d))$ and

$$\|D^\varepsilon\|_{L^\infty([0,T];W)} \leqslant \int_0^T \|b(t, \cdot)\|_W \, dt.$$

2. Assume $\omega \neq |\kappa|^2$, and $\partial_t b, \Delta b \in L^\infty([0, T]; W)$. Then we have the control

$$\|D^\varepsilon\|_{L^\infty([0,T];W)} \leqslant \frac{C\varepsilon}{||\kappa|^2 - \omega|} \left(\|b\|_{L^\infty([0,T];W)} + \|\Delta b\|_{L^\infty([0,T];W)} + \|\partial_t b\|_{L^\infty([0,T];W)} \right),$$

where C is independent of κ, ω and b.

Proof. By the definition of $U^\varepsilon(t)$, we have

$$\widehat{D}^\varepsilon(t, \xi) = \int_0^t e^{-i\varepsilon(t-\tau)|\xi|^2/2} \widehat{b} \left(\tau, \xi - \frac{\kappa}{\varepsilon} \right) e^{-i\omega\tau/(2\varepsilon)} d\tau.$$

Setting $\eta = \xi - \kappa/\varepsilon$, we have

$$\widehat{D}^\varepsilon(t, \xi) = e^{-i\varepsilon t|\eta + \kappa/\varepsilon|^2/2} \int_0^t e^{i\varepsilon\tau|\eta + \kappa/\varepsilon|^2/2} \widehat{b}(\tau, \eta) \, e^{-i\omega\tau/(2\varepsilon)} d\tau$$

$$= e^{-i\varepsilon t|\eta + \kappa/\varepsilon|^2/2} \int_0^t e^{i\tau\theta/2} \widehat{b}(\tau, \eta) \, d\tau,$$

where we have denoted

$$\theta = \varepsilon \left| \eta + \frac{\kappa}{\varepsilon} \right| - \frac{\omega}{\varepsilon} = \underbrace{\varepsilon|\eta|^2 + 2\kappa \cdot \eta}_{\theta_1} + \underbrace{\frac{|\kappa|^2 - \omega}{\varepsilon}}_{\theta_2}.$$

The first point of the lemma is straightforward. To prove the second point, integrate by parts, by first integrating $e^{i\tau\theta_2/2}$:

$$\widehat{D}^\varepsilon(t, \xi) = -\frac{2i}{\theta_2} e^{i\tau\theta/2} \widehat{b}(\tau, \eta) \Big|_0^t + \frac{2i}{\theta_2} \int_0^t e^{i\tau\theta/2} \left(i\frac{\theta_1}{2} \widehat{b}(\tau, \eta) + \widehat{\partial_t b}(\tau, \eta) \right) d\tau.$$

We infer, if $b, \partial_t b, \Delta b \in L^\infty([0, T]; W)$:

$$\sup_{t \in [0,T]} \|\widehat{D}^\varepsilon(t)\|_{L^1} \lesssim \frac{1}{|\theta_2|} \left(\|\widehat{b}\|_{L^\infty([0,T];L^1)} + \|\widehat{\Delta b}\|_{L^\infty([0,T];L^1)} + \|\widehat{\partial_t b}\|_{L^\infty([0,T];L^1)} \right).$$

This yields the second point of the lemma. □

5.3. **Construction of the exact solution.** As a preliminary step in establishing a WKB approximation we first need to know that (1.6) is well posed on $W(\mathcal{M})$.

Proposition 5.8. *Consider the initial value problem*

$$(5.2) \qquad i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = \lambda\varepsilon|u^\varepsilon|^{2\sigma} u^\varepsilon \quad ; \quad u^\varepsilon(0, x) = u_0^\varepsilon(x),$$

where $\sigma \in \mathbb{N}^*$, $\lambda \in \mathbb{R}$, and $x \in \mathcal{M}$, with either $\mathcal{M} = \mathbb{R}^d$, or $\mathcal{M} = \mathbb{T}^d$, in which case $\varepsilon^{-1} \in \mathbb{N}^*$. If $u_0^\varepsilon \in W(\mathcal{M})$, then there exists $T^\varepsilon > 0$ and a unique solution $u^\varepsilon \in C([0, T^\varepsilon]; W(\mathcal{M}))$ to (5.2).

Remark 5.9. At this stage, the dependence of T^ε upon ε is unknown. In particular, T^ε might go to zero as $\varepsilon \to 0$. The proof below actually shows that if u_0^ε is uniformly bounded in $W(\mathcal{M})$ for $\varepsilon \in]0, 1]$, then $T^\varepsilon > 0$ can be chosen independent of ε. This case includes initial data (1.7) which we consider for the WKB analysis.

Proof. Duhamel's formulation of (5.2) reads

$$u^\varepsilon(t) = U^\varepsilon(t)u_0^\varepsilon - i\lambda \int_0^t U^\varepsilon(t - \tau)\left(|u^\varepsilon|^{2\sigma}u^\varepsilon(\tau)\right)d\tau.$$

Denote by $\Phi^\varepsilon(u^\varepsilon)(t)$ the right hand side in the above formula. From Lemmae 5.5, 5.6 and 5.7, we have:

$$\|\Phi^\varepsilon(u^\varepsilon)(t)\|_W \leqslant \|u_0^\varepsilon\|_W + |\lambda| \int_0^t \|u^\varepsilon(\tau)\|_W^{2\sigma+1}d\tau,$$

and if $\|u^\varepsilon\|_{L^\infty([0,T];W)}, \|\widetilde{u}^\varepsilon\|_{L^\infty([0,T];W)} \leqslant R$, then there exists $C = C(R)$ such that

$$\|\Phi^\varepsilon(u^\varepsilon)(t) - \Phi^\varepsilon(\widetilde{u}^\varepsilon)(t)\|_W \leqslant C(R) \int_0^t \|u^\varepsilon(\tau) - \widetilde{u}^\varepsilon(\tau)\|_W \, d\tau, \quad \forall t \in [0, T].$$

A fixed point argument in

$$\left\{ u \in C([0, T]; W(\mathcal{M})), \ \sup_{t \in [0,T]} \|u(t)\|_W \leqslant 2\|u_0^\varepsilon\|_W \right\}$$

for $T = T^\varepsilon > 0$ sufficiently small then yields Proposition 5.8. $\qquad\square$

5.4. **Construction of the profiles.** In order to justify our multiphase WKB analysis, we first need to establish an existence theory for the system of profile equations. To this end, for all $\sigma \in \mathbb{N}^*$, we rewrite the system (4.2) in its integral form:

$$(5.3) \quad \forall j \in J, \quad a_j(t, x) = a_j(0, x - t\kappa_j) - i\lambda \int_0^t \mathcal{N}_\sigma(a, \dots, a)(\tau, x + (\tau - t)\kappa_j)d\tau,$$

where, for $a^{(1)} = (a_j^{(1)})_{j \in J}, \dots, a^{(2\sigma+1)} = (a_j^{(2\sigma+1)})_{j \in J}$, we define the nonlinear term $\mathcal{N}_\sigma$ by:

$$\forall j \in J, \quad \mathcal{N}_\sigma\left(a^{(1)}, \dots, a^{(2\sigma+1)}\right) = \sum_{(\ell_1, \dots, \ell_{2\sigma+1}) \in I_j} a_{\ell_1}^{(1)}\overline{a}_{\ell_2}^{(2)} \dots \overline{a}_{\ell_{2\sigma}}^{(2\sigma)} a_{\ell_{2\sigma+1}}^{(2\sigma+1)}.$$

It is clearly linear with respect to its arguments with odd exponents, and anti-linear with respect to the others. We prove in Lemma 5.11 below that it is in fact well defined and continuous on $E(\mathcal{M})$, for $\mathcal{M} = \mathbb{T}^d$ or $\mathcal{M} = \mathbb{R}^d$:

Definition 5.10. *Define*

$$E(\mathbb{R}^d) = \{a = (a_j)_{j \in J} \mid (\widehat{a}_j)_{j \in J} \in \ell^1(J; L^1(\mathbb{R}^d))\},$$

equipped with the norm

$$\|a\|_{E(\mathbb{R}^d)} = \sum_{j \in J} \|\widehat{a}_j\|_{L^1}.$$

Set also $E(\mathbb{T}^d) = \ell^1(J)$, equipped with the usual norm

$$\|a\|_{E(\mathbb{T}^d)} = \sum_{j \in J} |a_j|.$$

Note that E simply represents, *via* an isometric correspondence, the family of coefficients of functions in $\mathbf{A}$ (up to the choice of the wave numbers κ_j in the case of $\mathbb{R}^d$):

$$f(x,y) = \sum_{j\in J} a_j(x)e^{i\kappa_j \cdot y} \in \mathbf{A}(\mathbb{R}^d) \text{ iff } a \in E(\mathbb{R}^d),$$

and then $\|a\|_E = \|f\|_{\mathbf{A}}$. The same holds for $\mathcal{M} = \mathbb{T}^d$.

Lemma 5.11. *Let* $\sigma \in \mathbb{N}^*$. *For* $\mathcal{M} = \mathbb{R}^d$ *or* $\mathcal{M} = \mathbb{T}^d$, *the nonlinear expression* $\mathcal{N}_\sigma$ *defines a continuous mapping from* $E(\mathcal{M})^{2\sigma+1}$ *to* $E(\mathcal{M})$, *and for all* $a^{(1)}, \ldots, a^{(2\sigma+1)} \in E(\mathcal{M})$

$$\left\| \mathcal{N}_\sigma\left(a^{(1)}, \ldots, a^{(2\sigma+1)}\right) \right\|_E \leqslant \|a^{(1)}\|_E \ldots \|a^{(2\sigma+1)}\|_E.$$

Proof. We consider the case $\mathcal{M} = \mathbb{R}^d$, since $\mathcal{M} = \mathbb{T}^d$ is even simpler. In order to bound

$$\left\| \mathcal{N}_\sigma\left(a^{(1)}, \ldots, a^{(2\sigma+1)}\right) \right\|_E =$$

$$= \sum_{j\in J} \left\| \sum_{(\ell_1,\ldots,\ell_{2\sigma+1})\in I_j} \mathcal{F}\left(a^{(1)}_{\ell_1}\right) * \mathcal{F}\left(\overline{a}^{(2)}_{\ell_2}\right) * \cdots * \mathcal{F}\left(a^{(2\sigma+1)}_{\ell_{2\sigma+1}}\right) \right\|_{L^1}$$

$$\leqslant \sum_{j\in J} \sum_{(\ell_1,\ldots,\ell_{2\sigma+1})\in I_j} \left\| \mathcal{F}\left(a^{(1)}_{\ell_1}\right) * \mathcal{F}\left(\overline{a}^{(2)}_{\ell_2}\right) * \cdots * \mathcal{F}\left(a^{(2\sigma+1)}_{\ell_{2\sigma+1}}\right) \right\|_{L^1},$$

we use Young's inequality and observe that, once j, ℓ_1, $\ldots$, $\ell_{2\sigma}$ are chosen, $\ell_{2\sigma+1}$ is determined (since $\kappa_{\ell_{2\sigma+1}} = \kappa_j - \sum_{k=1}^{2\sigma}(-1)^{k+1}\kappa_{\ell_k}$, and $n \neq m \Rightarrow \kappa_n \neq \kappa_m$), so that

$$\left\| \mathcal{N}_\sigma\left(a^{(1)}, \ldots, a^{(2\sigma+1)}\right) \right\|_E \leqslant \sum_{(\ell_1,\ldots,\ell_{2\sigma+1})\in J^{2\sigma+1}} \left\| \mathcal{F}\left(a^{(1)}_{\ell_1}\right) \right\|_{L^1} \cdots \left\| \mathcal{F}\left(a^{(2\sigma+1)}_{\ell_{2\sigma+1}}\right) \right\|_{L^1},$$

which gives the desired result. $\qquad\square$

This consequently yields the following existence result for (5.3), where here and in the following we denote $\langle \kappa \rangle^2 \equiv 1 + |\kappa|^2$.

Proposition 5.12. *Let* $\sigma \in \mathbb{N}^*$, *and* $\mathcal{M} = \mathbb{R}^d$ *or* $\mathcal{M} = \mathbb{T}^d$.
For all $\alpha = (\alpha_j)_{j\in J} \in E(\mathcal{M})$, *there exist* $T > 0$ *and a unique solution*

$$t \mapsto a(t) = (a_j(t))_{j\in J} \in C([0,T], E(\mathcal{M}))$$

to the system (5.3), *with* $a(0) = \alpha$. *Moreover, the following properties hold:*
1. If $(\langle\kappa_j\rangle^s \alpha_j)_{j\in J} \in E(\mathcal{M})$ *for some* $s \in \mathbb{N}$, *then* $(\langle\kappa_j\rangle^s a_j)_{j\in J} \in C([0,T], E(\mathcal{M}))$.
2. On $\mathcal{M} = \mathbb{R}^d$, *if* $(\langle\kappa_j\rangle^s \partial_x^\beta \alpha_j)_{j\in J} \in E(\mathbb{R}^d)$, *for some* $\beta \in \mathbb{N}^d$ *and* $s \in \mathbb{N}$, *then* $(\langle\kappa_j\rangle^s \partial_x^\beta a_j)_{j\in J} \in C([0,T]; E(\mathbb{R}^d))$.

Proof. The existence result follows from Lemma 5.11 and the standard Cauchy–Lipschitz result for ODE's. Concerning the propagation of moments $\langle\kappa_j\rangle^s a_j$, we again apply a fixed-point argument, estimating nonlinear terms

$$\langle\kappa_j\rangle^s \mathcal{N}_\sigma(a^{(1)}, \ldots, a^{(2\sigma+1)})$$

as in the proof of Lemma 5.11, *via*

$$\langle \kappa_j \rangle^2 \equiv 1 + |\kappa_j|^2 = 1 + \sum_{k=1}^{2\sigma+1} (-1)^{k+1} |\kappa_{\ell_k}|^2$$

$$\leqslant \sum_{k=1}^{2\sigma+1} \langle \kappa_{\ell_k} \rangle^2 \leqslant (2\sigma+1) \prod_{k=1}^{2\sigma+1} \langle \kappa_{\ell_k} \rangle^2,$$

when $(\ell_1, \ldots, \ell_{2\sigma+1}) \in I_j$. The last statement of the proposition is concerned with the smooth dependence upon the parameter x. This follows by commuting (4.2) with ∂_x and using the fact that $W(\mathbb{R}^d)$ is an algebra, continuously embedded in L^∞, since then

$$\frac{d}{dt} \|\partial_x a\|_E \lesssim \|\partial_x \alpha\|_E + C(\|a\|_E) \|\partial_x a\|_E,$$

and a Gronwall argument shows that $\|\partial_x a\|_E$ remains bounded for all $t \in [0, T]$. Similarly we conclude for the higher order derivatives, possibly multiplied by weights $\langle \kappa_j \rangle^s$. $\qquad \square$

For the particular situation for $\sigma = 1$, in $d = 1$ and/or the case of only two initial phases, we infer a stronger result, thanks to the explicit formulas given in §3.1 and §3.2.

Corollary 5.13. *Under the assumption of Proposition 5.12, in the case $\sigma = 1$, if in addition $d = 1$, then T can be taken arbitrarily large, with $a_j(t)$ explicitly given by (3.4) and (3.5). Similarly, if $\sharp J_0 \leqslant 2$, then T can be taken arbitrarily large.*

Remark 5.14. In the case of higher order nonlinearities, *i.e.* $\sigma \geqslant 2$, Equation (4.3) makes it possible to see, *via* explicit integration (see (4.4) in the case of the torus), that if $\alpha_j, \alpha_\ell \in W(\mathcal{M})$, then $a_j, a_\ell \in C([0, \infty[, W(\mathcal{M}))$.

6. Rigorous justification of the multiphase WKB analysis

6.1. **Construction of an approximate solution.** We start from oscillating initial data, given by a profile in $\mathbf{A}(\mathcal{M})$, with $\mathcal{M} = \mathbb{T}^d$ or $\mathbb{R}^d$:

$$u_{\text{app}}^\varepsilon(0, x) = \sum_{j \in J_0} \alpha_j(x) e^{i\kappa_j \cdot x/\varepsilon},$$

with $\alpha_j(x) = \text{Const.}$ in the case $\mathcal{M} = \mathbb{T}^d$.

Assumption 6.1. *For both $\mathcal{M} = \mathbb{R}^d$ and $\mathcal{M} = \mathbb{T}^d$ we assume $(\alpha_j)_{j \in J_0} \in E(\mathcal{M})$. For $\mathcal{M} = \mathbb{R}^d$ we assume in addition*

$$\forall |\beta| \leqslant 2, \ (\partial_x^\beta \alpha_j)_{j \in J_0} \in E(\mathbb{R}^d), \quad and \quad \forall |\beta| \leqslant 1, \ (\langle \kappa_j \rangle \partial_x^\beta \alpha_j)_{j \in J_0} \in E(\mathbb{R}^d).$$

From Proposition 5.12 we know, that these data produce a solution $(a_j)_{j \in J} \in C([0, T], E(\mathcal{M}))$ to the amplitude system and we consequently define the approximate solution $u_{\text{app}}^\varepsilon$ by

$$(6.1) \qquad u_{\text{app}}^\varepsilon(t, x) = \sum_{j \in J} a_j^\varepsilon(t, x) e^{i\phi_j(t,x)/\varepsilon},$$

with ϕ_j given by (2.1). The sequence $(a_j)_{j \in J}$ is such that

$$\left(\partial_x^\beta a_j \right)_{j \in J} \in C([0, T], E(\mathcal{M})), \quad |\beta| \leqslant 2,$$

$$\left(\langle \kappa_j \rangle \partial_x^\beta a_j \right)_{j \in J} \in C([0, T], E(\mathcal{M})), \quad |\beta| \leqslant 1.$$

We see from equation (5.3) that $(\partial_t a_j)_{j \in J} \in C([0,T], E(\mathcal{M}))$. We find (in the sense of distributions)

$$i\varepsilon \partial_t u^\varepsilon_{\text{app}} + \frac{\varepsilon^2}{2} \Delta u^\varepsilon_{\text{app}} = \lambda\varepsilon |u^\varepsilon_{\text{app}}|^{2\sigma} u^\varepsilon_{\text{app}} - \lambda\varepsilon r^\varepsilon_1 + \varepsilon^2 r^\varepsilon_2,$$

where

$$(6.2) \qquad r^\varepsilon_2 = \frac{1}{2} \sum_{j \in J} e^{i\phi_j/\varepsilon} \Delta a_j,$$

and the remainder r^ε_1 takes into account the non-characteristic phases created by nonlinear interaction. This means that it is a sum of terms of the form

$$a_{\ell_1} \overline{a}_{\ell_2} \ldots \overline{a}_{\ell_{2\sigma}} a_{\ell_{2\sigma+1}} e^{i(\phi_{\ell_1} - \phi_{\ell_2} + \cdots - \phi_{\ell_{2\sigma}} + \phi_{\ell_{2\sigma+1}})/\varepsilon},$$

where the rapid phase is given by

$$\sum_{p=1}^{2\sigma+1} (-1)^{p+1} \phi_{\ell_p}(t,x) = \left(\sum_{p=1}^{2\sigma+1} (-1)^{p+1} \kappa_{\ell_p} \right) \cdot x - \frac{t}{2} \sum_{p=1}^{2\sigma+1} (-1)^{p+1} |\kappa_{\ell_p}|^2,$$

and

$$\left| \sum_{p=1}^{2\sigma+1} (-1)^{p+1} \kappa_{\ell_p} \right|^2 \neq \sum_{p=1}^{2\sigma+1} (-1)^{p+1} \left| \kappa_{\ell_p} \right|^2.$$

In other words, $(\ell_1, \ldots, \ell_{2\sigma+1})$ belongs to the *non-resonant set*

$$N := J^{2\sigma+1} \setminus \bigcup_{j \in J} I^\sigma_j.$$

With these conventions, we have

$$(6.3) \qquad r^\varepsilon_1 = \sum_{(\ell_1, \ldots, \ell_{2\sigma+1}) \in N} a_{\ell_1} \overline{a}_{\ell_2} \ldots a_{\ell_{2\sigma+1}} e^{i(\phi_{\ell_1} - \phi_{\ell_2} + \cdots - \phi_{\ell_{2\sigma}} + \phi_{\ell_{2\sigma+1}})/\varepsilon}.$$

Estimating r^ε_2 in W is straightforward, since $(\partial^\beta_x a_j)_{j \in J} \in C([0,T], E)$ for $|\beta| \leqslant 2$:

$$(6.4) \qquad \|r^\varepsilon_2\|_W \leqslant \frac{1}{2} \|\Delta a\|_E.$$

Note that r_2 simply vanishes if $\mathcal{M} = \mathbb{T}^d$. In order to estimate r^ε_1, we impose the following condition on the set of wave numbers $\{\kappa_j\}_{j \in J}$.

Assumption 6.2. *There exists $c > 0$ such that for all $(\ell_1, \ldots, \ell_{2\sigma+1}) \in N$,*

$$\delta(\ell_1, \ldots, \ell_{2\sigma+1}) \equiv \left| \left| \sum_{p=1}^{2\sigma+1} (-1)^{p+1} \kappa_{\ell_p} \right|^2 - \sum_{p=1}^{2\sigma+1} (-1)^{p+1} \left| \kappa_{\ell_p} \right|^2 \right| \geqslant c.$$

Remark 6.3. (i) This assumption is of course satisfied when only finitely many phases are created $\sharp J < \infty$.
(ii) Similarly, this assumption holds for $\{\kappa_j\}_{j \in J} \subset \mathbb{Z}^d$, since in this case, the quantity considered is an integer.
(iii) Consider the cubic case $\sigma = 1$, and suppose that $\{\kappa_j\}_{j \in J}$ is included in a rectangular net. Up to translation, this rectangular net has the form

$$\{Am \in \mathbb{R}^d \mid m \in \mathbb{Z}^d\},$$

with A a $d \times d$ matrix of the form $A = RD$, where D is diagonal, and R is a rotation. Then we have, for all $k, l, m \in \mathbb{Z}^d$:

$$\left| |Ak - A\ell + Am|^2 - |Ak|^2 + |A\ell|^2 - |Am|^2 \right| = |(Ak - A\ell) \cdot (Ak - Am)|$$
$$= \left| (k - l) \cdot \left((A^T A)(k - m) \right) \right|.$$

Since $^T AA = D^2$, denoting $\mu_1^2, \ldots, \mu_d^2$ the squares of the eigenvalues of D, Assumption 6.2 is then satisfied if and only if the group generated by $\mu_1^2, \ldots, \mu_d^2$ in $\mathbb{R}$ is discrete, *i.e.* these numbers are (pairwise) rationally dependent.

The reason for imposing the above assumption is a small divisor problem, as will become clear from the proof of the following lemma. It is possible to relax Assumption 6.2 to a less rigid one, to the cost of a more technical presentation. The latter is sketched in an appendix.

Lemma 6.4. *For $\mathcal{M} = \mathbb{T}^d$ or $\mathcal{M} = \mathbb{R}^d$, let r_1^ε be given by (6.3) and denote*

$$(6.5) \qquad R_1^\varepsilon(t, x) := \int_0^t U^\varepsilon(t - \tau) r_1^\varepsilon(\tau, x) d\tau, \quad \text{on } [0, T] \times \mathcal{M}.$$

Let Assumptions 6.1–6.2 hold. Then, there exists a constant $C > 0$, such that:

$$\|R_1^\varepsilon\|_{L^\infty([0,T]; W(\mathcal{M}))} \leqslant C\varepsilon.$$

Proof. We only treat the case on $\mathcal{M} = \mathbb{R}^d$ in detail. The case $\mathcal{M} = \mathbb{T}^d$ can be treated analogously. We have

$$R_1^\varepsilon(t, x) =$$
$$\sum_{(\ell_1, \ldots, \ell_{2\sigma+1}) \in N} \int_0^t U^\varepsilon(t - \tau) \left((a_{\ell_1} \overline{a}_{\ell_2} \ldots a_{\ell_{2\sigma+1}}) e^{i(\phi_{\ell_1} - \phi_{\ell_2} + \cdots + \phi_{\ell_{2\sigma+1}})/\varepsilon} \right) (\tau, x) \, d\tau.$$

Thus, setting $b_{\ell_1, \ldots, \ell_{2\sigma+1}} := a_{\ell_1} \overline{a}_{\ell_2} \ldots a_{\ell_{2\sigma+1}}$, Lemma 5.7 yields

$$\|R_1^\varepsilon\|_{L^\infty([0,T]; W)} \lesssim \varepsilon \sum_{(\ell_1, \ldots, \ell_{2\sigma+1}) \in N} \frac{1}{\delta(\ell_1, \ldots, \ell_{2\sigma+1})} \Big(\|\widehat{b}_{\ell_1, \ldots, \ell_{2\sigma+1}}\|_{L^\infty([0,T]; L^1)}$$
$$+ \|\widehat{\Delta b}_{\ell_1, \ldots, \ell_{2\sigma+1}}\|_{L^\infty([0,T]; L^1)} + \|\widehat{\partial_t b}_{\ell_1, \ldots, \ell_{2\sigma+1}}\|_{L^\infty([0,T]; L^1)} \Big)$$
$$\lesssim \varepsilon \sum_{(\ell_1, \ldots, \ell_{2\sigma+1}) \in N} \Big(\|\widehat{b}_{\ell_1, \ldots, \ell_{2\sigma+1}}\|_{L^\infty([0,T]; L^1)}$$
$$+ \|\widehat{\Delta b}_{\ell_1, \ldots, \ell_{2\sigma+1}}\|_{L^\infty([0,T]; L^1)} + \|\widehat{\partial_t b}_{\ell_1, \ldots, \ell_{2\sigma+1}}\|_{L^\infty([0,T]; L^1)} \Big),$$

where we have used Assumption 6.2. Next, using Young's inequality, as in the proof of Lemma 5.11, we get:

$$\sum_{(\ell_1, \ldots, \ell_{2\sigma+1}) \in N} \|\widehat{b}_{\ell_1, \ldots, \ell_{2\sigma+1}}\|_{L^\infty([0,T]; L^1)} \lesssim \sum_{(\ell_1, \ldots, \ell_{2\sigma+1}) \in N} \|\widehat{a}_{\ell_1}\|_{L_T^\infty L^1} \ldots \|\widehat{a}_{\ell_{2\sigma+1}}\|_{L_T^\infty L^1}$$
$$\lesssim \sum_{(\ell_1, \ldots, \ell_{2\sigma+1}) \in J^{2\sigma+1}} \|\widehat{a}_{\ell_1}\|_{L_T^\infty L^1} \ldots \|\widehat{a}_{\ell_{2\sigma+1}}\|_{L_T^\infty L^1}$$
$$\lesssim \|(a_j)_{j \in J}\|_{L^\infty([0,T]; E)}^{2\sigma+1}.$$

Leibniz formula and Hölder inequality yield similar estimates for $\widehat{\Delta b}_{\ell_1, \ldots, \ell_{2\sigma+1}}$ and $\widehat{\partial_t b}_{\ell_1, \ldots, \ell_{2\sigma+1}}$ in $L^\infty([0, T]; L^1(\mathcal{M}))$, and the lemma follows. $\square$

6.2. **Accuracy of the multiphase WKB approximation.** With the above results in hand, we can now prove our main theorem.

Theorem 6.5 (General approximation result)**.** *Let $\sigma \geqslant 1$, $\mathcal{M} = \mathbb{T}^d$ or $\mathbb{R}^d$, and Assumptions 6.1–6.2 hold. Given an approximate solution $u_{\mathrm{app}}^\varepsilon \in C([0,T]; W(\mathcal{M}))$ as in (6.1), we consider a family of initial data $(u_0^\varepsilon)_{\varepsilon>0} \in W(\mathcal{M})$, such that*

$$\left\| u_0^\varepsilon - u_{\mathrm{app|t=0}}^\varepsilon \right\|_{W(\mathcal{M})} \leqslant C_0 \varepsilon,$$

for some $C_0 \geqslant 0$ independent of ε. Then there exists $\varepsilon_0(T) > 0$, such that for any $0 < \varepsilon \leqslant \varepsilon_0(T)$, the exact solution to the Cauchy problem (5.2) satisfies $u^\varepsilon \in L^\infty([0,T]; W(\mathcal{M}))$. In addition, $u_{\mathrm{app}}^\varepsilon$ approximates u^ε up to $\mathcal{O}(\varepsilon)$:

$$\left\| u^\varepsilon - u_{\mathrm{app}}^\varepsilon \right\|_{L^\infty([0,T]\times\mathcal{M})} \leqslant \left\| u^\varepsilon - u_{\mathrm{app}}^\varepsilon \right\|_{L^\infty([0,T];W(\mathcal{M}))} \leqslant C\varepsilon,$$

where C is independent of ε.

Obviously the result for $x \in \mathbb{T}$, announced in the introduction, can be seen as a special case of Theorem 6.5.

Proof. From Proposition 5.8, we may consider a solution $u^\varepsilon \in C([0,T^\varepsilon], W(\mathcal{M}))$ to (1.6). We define the difference $w^\varepsilon := u^\varepsilon - u_{\mathrm{app}}^\varepsilon$. Then $w^\varepsilon \in C([0,\tau^\varepsilon], W(\mathcal{M}))$, where $\tau^\varepsilon = \min(T^\varepsilon, T)$. We prove that for ε sufficiently small, w^ε may be extended up to time T, with $w^\varepsilon \in C([0,T], W(\mathcal{M}))$. Take $\varepsilon_0 > 0$ so that $C_0\varepsilon_0 \leq 1/2$, and for $\varepsilon \in]0, \varepsilon_0]$, let

$$t^\varepsilon := \sup \left\{ t \in [0,T] \mid \sup_{t'\in[0,t]} \|w^\varepsilon(t')\|_{W(\mathcal{M})} \leqslant 1 \right\}.$$

We already know that $t^\varepsilon > 0$ by the local existence result for u^ε. By possibly reducing $\varepsilon_0 > 0$, we shall show that $t^\varepsilon \geqslant T$. The error term w^ε solves:

$$i\partial_t w^\varepsilon + \frac{\varepsilon}{2}\Delta w^\varepsilon = \lambda\left(|u_{\mathrm{app}}^\varepsilon + w^\varepsilon|^{2\sigma}(u_{\mathrm{app}}^\varepsilon + w^\varepsilon) - |u_{\mathrm{app}}^\varepsilon|^{2\sigma}u_{\mathrm{app}}^\varepsilon\right) + \lambda r_1^\varepsilon - \varepsilon r_2^\varepsilon,$$

where r_1^ε, r_2^ε are given in (6.2)–(6.3). Using Duhamel's formula we can rewrite this equation as

$$w^\varepsilon(t) = U^\varepsilon(t)w_0^\varepsilon - i\lambda \int_0^t U^\varepsilon(t-\tau)\left(|u_{\mathrm{app}}^\varepsilon + w^\varepsilon|^{2\sigma}(u_{\mathrm{app}}^\varepsilon + w^\varepsilon) - |u_{\mathrm{app}}^\varepsilon|^{2\sigma}u_{\mathrm{app}}^\varepsilon\right)(\tau)\,d\tau$$

$$- i\lambda R_1^\varepsilon(t) + i\varepsilon \int_0^t U^\varepsilon(t-\tau)r_2^\varepsilon(\tau)\,d\tau,$$

where R_1^ε is defined in (6.5). Using the fact that $U^\varepsilon(t)$ is unitary on $W(\mathcal{M})$, and the estimates given in (6.4) and in Lemma 6.4, we obtain on $[0, t^\varepsilon]$:

$$\|w^\varepsilon(t)\|_{W(\mathcal{M})} \leqslant C_1\varepsilon + |\lambda| \int_0^t \|\left(|u_{\mathrm{app}}^\varepsilon + w^\varepsilon|^{2\sigma}(u_{\mathrm{app}}^\varepsilon + w^\varepsilon) - |u_{\mathrm{app}}^\varepsilon|^{2\sigma}u_{\mathrm{app}}^\varepsilon\right)(\tau)\|_{W(\mathcal{M})}\,d\tau$$

$$\leqslant C_1\varepsilon + C_2 \int_0^t \|w^\varepsilon(\tau)\|_{W(\mathcal{M})}\,d\tau,$$

by the Lipschitz property from Lemma 5.5 . Note that, in view of Lemma 5.2 , resp. Lemma 5.4, $(u_{\mathrm{app}}^\varepsilon)_{\varepsilon>0}$ is a bounded family in $C([0,T], W(\mathcal{M}))$, and restricting t to

$[0, t^\varepsilon]$ ensures that $w^\varepsilon(t)$ stays bounded in $W(\mathcal{M})$. The constants C_1, C_2 depend on C_0 and $u^\varepsilon_{\mathrm{app}}$. Now, Gronwall lemma yields

$$\|w^\varepsilon(t)\|_{W(\mathcal{M})} \leqslant C_1\varepsilon\left(1 + \frac{e^{C_2 T}}{C_2}\right),$$

and we may reduce ε_0 so that $C_1\varepsilon_0\left(1 + e^{C_2 T}/C_2\right) < 1$. This shows that $t^\varepsilon \geqslant T$, for all $\varepsilon \in]0, \varepsilon_0]$. Then, $T^\varepsilon \geqslant T$ follows, as well as the desired approximation of u^ε by $u^\varepsilon_{\mathrm{app}}$, since $w^\varepsilon = \mathcal{O}(\varepsilon)$ in $L^\infty([0,T];W)$. $\qquad\square$

7. Proof of the instability result

This section is devoted to the proof of Theorem 1.2. To this end we essentially rewrite the proof of M. Christ, J. Colliander, and T. Tao [8] in terms of weakly nonlinear geometric optics. It then becomes easy to see that the justification given in the previous paragraph makes it possible to extend the one-dimensional analysis of [8] in order to infer Theorem 1.2.

Proof of Theorem 1.2. We start with two Fourier modes, one of them being zero:

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda|u|^{2\sigma}u \quad ; \quad u(0,x) = \alpha_0 + \alpha_1 e^{iKx_1}, \quad K \in \mathbb{N}.$$

The fact that we privilege oscillations with respect to the first space variable is purely arbitrary. Define $\widetilde{u}$ as the solution to the same equation, with data

$$\widetilde{u}(0,x) = \widetilde{\alpha}_0 + \widetilde{\alpha}_1 e^{iKx_1}.$$

Let

$$\varepsilon = \frac{1}{K^2} \quad ; \quad u^\varepsilon(t,x) = u\left(t, \frac{x}{\sqrt{\varepsilon}}\right) = u\left(t, Kx\right).$$

(ε is chosen so that we remain on the torus.) We see that u^ε solves (1.6) on $\mathbb{T}^d$, with

$$u^\varepsilon(0,x) = \alpha_0 + \alpha_1 e^{ix_1/\varepsilon}.$$

From Theorem 6.5, we know that there exists $T > 0$ independent of ε, such that

$$\|u^\varepsilon - u^\varepsilon_{\mathrm{app}}\|_{L^\infty([0,T]\times\mathbb{T}^d)} + \|\widetilde{u}^\varepsilon - \widetilde{u}^\varepsilon_{\mathrm{app}}\|_{L^\infty([0,T]\times\mathbb{T}^d)} = \mathcal{O}(\varepsilon),$$

where $u^\varepsilon_{\mathrm{app}}$ is the approximate solution defined by (6.1), and $\widetilde{u}^\varepsilon_{\mathrm{app}}$ is defined similarly. On the other hand, we have

$$u^\varepsilon_{\mathrm{app}}(t,x) = \alpha_0 e^{-i\lambda t\theta_0} + \alpha_1 e^{-i\lambda t\theta_1} e^{i(x_1 - t/2)/\varepsilon},$$

where, in view of (4.3), θ_0 is given by

$$\theta_0 = \sum_{n=0}^{\sigma}\binom{\sigma+1}{n}\binom{\sigma}{n}|\alpha_0|^{2\sigma-2nl}|\alpha_1|^{2n}.$$

We infer, uniformly in $t \in [0,T]$,

$$\left|\int_{\mathbb{T}^d}(u(t,x) - \widetilde{u}(t,x))dx\right| = \left|\alpha_0 e^{-i\lambda t\theta_0} - \widetilde{\alpha}_0 e^{-i\lambda t\widetilde{\theta}_0}\right| + \mathcal{O}(\varepsilon),$$

with obvious notations.

To prove the first point of Theorem 1.2, set

$$\alpha_0 = \widetilde{\alpha}_0 = \frac{\rho}{2}, \quad \alpha_1 = \frac{\rho}{2K^s} = \frac{\rho}{2}K^{|s|}, \quad \widetilde{\alpha}_1 = \sqrt{\alpha_1^2 + \frac{1}{\delta}}.$$

We infer, for $0 < \delta \leqslant 1$,

$$\left|\theta_0 - \widetilde{\theta}_0\right| \gtrsim \frac{1}{\delta}.$$

We have $\|u(0) - \widetilde{u}(0)\|_{H^s} < \delta$ provided $K > \delta^{1/s}$. Since $\widetilde{\alpha}_0 = \alpha_0$, we also have

$$\left|\int_{\mathbb{T}^d} (u(t,x) - \widetilde{u}(t,x))dx\right| = \left|2\alpha_0 \sin\left(\frac{\lambda t}{2}\left(\widetilde{\theta}_0 - \theta_0\right)\right)\right| + \mathcal{O}(\varepsilon).$$

We infer that we can find $t \in [0,\delta]$ so that the right hand side is bounded from below by $\rho/2$, provided N is sufficiently large (hence ε sufficiently small).

To prove the second point of Theorem 1.2, set

$$\alpha_0 = \frac{\rho}{2}, \quad \widetilde{\alpha}_0 = \alpha_0 + \delta, \quad \alpha_1 = \widetilde{\alpha}_1 = \frac{\rho}{2K^s} = \frac{\rho}{2}K^{|s|}.$$

For δ small compared to ρ, we use the same estimate as above,

$$\left|\int_{\mathbb{T}^d} (u(t,x) - \widetilde{u}(t,x))dx\right| \gtrsim \left|2\alpha_0 \sin\left(\frac{\lambda t}{2}\left(\widetilde{\theta}_0 - \theta_0\right)\right)\right|,$$

for K sufficiently large. We now have

$$\left|\theta_0 - \widetilde{\theta}_0\right| \gtrsim \left||\alpha_0|^{2\sigma} - |\widetilde{\alpha}_0|^{2\sigma}\right| + |\alpha_1|^{2\sigma-2}\left||\alpha_0|^2 - |\widetilde{\alpha}_0|^2\right|$$

$$\gtrsim \delta + \left(\rho K^{|s|}\right)^{2\sigma-2}\delta.$$

Now we see that if we assume $\sigma \geqslant 2$, the left hand side can be estimated from below by $1/\delta$, provided N is sufficiently large, and we conclude like for the first point.

To prove the last point in Theorem 1.2, we resume the argument of [25]. Fix $\alpha_0 \in \mathbb{C} \setminus \{0\}$, and let $\alpha_1 \in \mathbb{C}$ to be fixed later. As $K \to \infty$, we have:

$$u(0,\cdot) \rightharpoonup \alpha_0 =: \underline{u}(0,\cdot) \text{ weakly in } L^2(\mathbb{T}^d) \quad ; \quad \|u(0)\|_{L^2}^2 \to |\alpha_0|^2 + |\alpha_1|^2.$$

For any $t > 0$, we have, as $K \to \infty$,

$$u(t,x) \rightharpoonup \alpha_0 e^{-i\lambda t\theta_0} \text{ weakly in } L^2(\mathbb{T}^d),$$

where

$$\theta_0 = \sum_{n=0}^{\sigma}\binom{\sigma+1}{n}\binom{\sigma}{n}|\alpha_0|^{2\sigma-2n}|\alpha_1|^{2n}.$$

Note that for any $\alpha_0 \in \mathbb{C} \setminus \{0\}$ and any angle $\theta \in [0, 2\pi[$, we can find $\alpha_1 \in \mathbb{C}$ so that $\theta_0 = \theta + |\alpha_0|^{2\sigma}$. On the other hand, the solution to (1.8) with initial data α_0 is given by

$$\underline{u}(t,x) = \alpha_0 e^{-i\lambda t|\alpha_0|^{2\sigma}}.$$

We infer

$$w - \lim_{N\to\infty} u(t,x) - \underline{u}(t,x) = \alpha_0 e^{-i\lambda t|\alpha_0|^{2\sigma}}\left(e^{-i\lambda t\theta} - 1\right).$$

For all $t \neq 0$, one can then choose θ so that $\lambda t\theta \notin 2\pi\mathbb{Z}$. The discontinuity at α_0 of the map $\alpha_0 \mapsto \underline{u}(t)$, from $L^2(\mathbb{T}^d)$ equipped with its weak topology into $\left(C^\infty(\mathbb{T}^d)\right)^*$, follows. $\qquad\square$

Appendix A. A more general set of initial phases

We can actually replace Assumption 6.2 with the following more general one:

Assumption A.1. *There exist $b \geqslant 0$, $c > 0$ such that for all $(\ell_1, \ldots, \ell_{2\sigma+1}) \in N$,*

$$\delta\left(\ell_1, \ldots, \ell_{2\sigma+1}\right) \equiv \left| \left| \sum_{p=1}^{2\sigma+1} (-1)^{p+1} \kappa_{\ell_p} \right|^2 - \sum_{p=1}^{2\sigma+1} (-1)^{p+1} \left| \kappa_{\ell_p} \right|^2 \right|$$

satisfies:

$$\delta(\ell_1, \ldots, \ell_{2\sigma+1}) \geqslant c \left\langle \kappa_{\ell_1} \right\rangle^{-b} \ldots \left\langle \kappa_{\ell_{2\sigma+1}} \right\rangle^{-b}.$$

In §6, we have considered the case $b = 0$. However, allowing constants $b > 0$, we show that the assumption is satisfied by wave vector sets included in generic finitely generated nets.

Proposition A.2. *For all $p \in \mathbb{N}^*$, there exist $C, b > 0$ and $Z \subset \mathbb{R}^{dp}$ with zero Lebesgue measure such that, for all $(\kappa_1, \ldots, \kappa_p) \in \mathbb{R}^{dp} \setminus Z$, the set $(\kappa_j)_{j \in J}$ constructed from these initial wave vectors $\{\kappa_j\}_{j \in J_0}$ satisfies Assumption A.1.*

Proof. We shall prove that the above result holds when Assumption A.1 is replaced by the stronger one, where N is replaced by $J^{2\sigma+1}$.

All the wave vectors we consider belong to the group generated by $\{\kappa_1, \ldots, \kappa_p\}$. Thus, to each $\ell_k \in J$ corresponds $(\alpha_{k,1}, \ldots, \alpha_{k,p}) \in \mathbb{Z}^p$, such that: $\kappa_{\ell_k} = \alpha_{k,1}\kappa_1 + \cdots + \alpha_{k,p}\kappa_p$. With this notation, for all $(\ell_1, \ldots, \ell_{2\sigma+1}) \in J^{2\sigma+1}$, we have:

$$\begin{aligned}
\delta(\ell_1, \ldots, \ell_{2\sigma+1}) &= \left| \left| \sum_{k=1}^{2\sigma+1} (-1)^{k+1} \sum_{j=1}^{p} \alpha_{k,j}\kappa_j \right|^2 + \sum_{m=1}^{2\sigma+1} (-1)^{m+1} \left| \sum_{j=1}^{p} \alpha_{m,j}\kappa_j \right|^2 \right| \\
&= \left| \sum_{i,j=1}^{p} \left(\sum_{k,\ell=1}^{2\sigma+1} (-1)^{k+\ell} \alpha_{k,i}\alpha_{\ell,j} - \sum_{m=1}^{2\sigma+1} (-1)^m \alpha_{m,i}\alpha_{m,j} \right) \kappa_i \cdot \kappa_j \right|.
\end{aligned}$$

Now, a standard Diophantine result (see *e.g.* [1, 11]) ensures that, for all choice of $(\kappa_i \cdot \kappa_j)_{1 \leqslant i,j \leqslant p}$ but in some subset of $\mathbb{R}^{p^2}$ with measure zero, we have, for some $b' \geqslant 0$ and $C' > 0$:

$$\forall (\beta_{i,j})_{1 \leqslant i,j \leqslant p} \in \mathbb{Z}^{p^2} \setminus \{0\}, \quad \left| \sum_{i,j=1}^{p} \beta_{i,j}\kappa_i \cdot \kappa_j \right| \geqslant C' \left(\sum_{i,j=1}^{p} |\beta_{i,j}| \right)^{-b'}.$$

Such an estimate is then valid for almost all $(\kappa_1, \ldots, \kappa_p)$ in $(\mathbb{R}^d)^p$. We apply it with

$$\beta_{i,j} = \sum_{k,\ell=1}^{2\sigma+1} (-1)^{k+\ell} \alpha_{k,i}\alpha_{\ell,j} - \sum_{m=1}^{2\sigma+1} (-1)^m \alpha_{m,i}\alpha_{m,j},$$

so that

$$\sum_{i,j=1}^{p} |\beta_{i,j}| \leqslant 2 \sum_{k,\ell=1}^{2\sigma+1} |\alpha_{k,\cdot}| |\alpha_{\ell,\cdot}|$$

$$\leqslant 2(2\sigma+1)^2 \prod_{k=1}^{p} \left\langle \alpha_{k,\cdot} \right\rangle^2.$$

Now, choosing $\kappa_1, \ldots, \kappa_p$ $\mathbb{Q}$-linearly independent (which is true almost surely), we get that there exists a constant $c > 0$ such that

$$\forall \alpha \in \mathbb{Q}^p, \quad |\alpha_1| + \cdots + |\alpha_p| \leqslant c \sum_{j=1}^{d} |(\alpha_1 \kappa_1 + \cdots + \alpha_p \kappa_p)_j|.$$

Increasing c if necessary, so that $c \geqslant 1$, we get, when $\kappa_{\ell_k} = \alpha_{k,1} \kappa_1 + \cdots + \alpha_{k,p} \kappa_p$: $\langle \alpha_{k,\cdot} \rangle \leqslant c \langle \kappa_{\ell_k} \rangle$. Finally, using the constants b' and C' from above, the desired estimate follows with $b = 2b'$ and $C = (2(2\sigma + 1)^2 c^2)^{-b'} C'$. $\qquad \square$

Under Assumption A.1 (which is fairly general for plane waves, in view of the above proposition), we can easily adapt the analysis of §6. Essentially, we have to (possibly) strengthen the assumptions on the initial profile, in the case of $\mathcal{M} = \mathbb{R}^d$, where we generalize Assumption 6.1 to:

Assumption A.3. *On $\mathcal{M} = \mathbb{R}^d$, the initial amplitudes satisfy:*

$$\forall |\beta| \leqslant 2, \ (\langle \kappa_j \rangle^b \partial_x^\beta \alpha_j)_{j \in J_0} \in E(\mathbb{R}^d),$$
$$\forall |\beta| \leqslant 1, \ (\langle \kappa_j \rangle^{1+b} \partial_x^\beta \alpha_j)_{j \in J_0} \in E(\mathbb{R}^d).$$

From Proposition 5.12, these data produce a solution $(a_j)_{j \in J} \in C([0,T], E(\mathcal{M}))$ to the profile system (5.3). We consequently define the approximate solution $u_{\mathrm{app}}^\varepsilon$ as before

$$u_{\mathrm{app}}^\varepsilon(t,x) = \sum_{j \in J} a_j^\varepsilon(t,x) e^{i\phi_j(t,x)/\varepsilon},$$

where the sequence $(a_j)_{j \in J}$ is now such that

$$\left(\langle \kappa_j \rangle^b \partial_x^\beta a_j \right)_{j \in J} \in C([0,T], E(\mathcal{M})), \quad |\beta| \leqslant 2,$$
$$\left(\langle \kappa_j \rangle^{1+b} \partial_x^\beta a_j \right)_{j \in J} \in C([0,T], E(\mathcal{M})), \quad |\beta| \leqslant 1.$$

We can then reproduce the analysis of §6: Lemma 6.4 is still valid under Assumption A.1 and A.3, by straightforward verification. Then one just has to notice that this is the only step where the absence of small divisors plays a role in the proof of Theorem 6.5. Therefore, Theorem 6.5 remains valid under Assumption A.1 and A.3.

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(R. Carles) Univ. Montpellier 2, Mathématiques, CC051, F-34095 Montpellier

CNRS, UMR 5149, Montpellier, F-34095
E-mail address: Remi.Carles@math.cnrs.fr

(E. Dumas) Univ. Grenoble 1, Institut Fourier, 100, rue des Mathématiques-BP 74, 38402 Saint Martin d'Hères cedex, France
E-mail address: Eric.Dumas@ujf-grenoble.fr

(C. Sparber) Department of Applied Mathematics and Theoretical Physics, CMS, Wilberforce Road, Cambridge CB3 0WA, England
E-mail address: c.sparber@damtp.cam.ac.uk